NASA-CR-202112

Finite Elements and Finite Differences For Transonic Flow Calculations

By

M. M. Hafez
L. C. Wellford
E. M. Murman

May 1977

Flow Research Company
A Division of Flow Industries, Inc.
P. O. Box 5040
Kent, Washington 98031
(206) 854-1370

FINITE ELEMENTS AND FINITE DIFFERENCES FOR TRANSONIC FLOW CALCULATIONS*

by

M. M. Hafez, L. C. Wellford**, and E. M. Murman

Flow Research Company, Kent, Washington 98031

Abstract

Finite difference methods for transonic flow calculations are briefly reviewed. Applications of the finite element approach are studied. Different discretization techniques, with different regularization methods as well as different iterative procedures for solving the approximate problem, are examined.

---

* This research was supported by NASA Langley Research Center Contract NAS1-14246 and partially by Wright Patterson Air Force Base Under Contract No. F33615-76-C-3067.

** Assistant Professor, University of Southern California, Los Angeles.

12.1 Introduction

Steady inviscid transonic flow is described by nonlinear partial differential equations whose types change from elliptic in subsonic regions to hyperbolic in supersonic regions of the domain. The boundary between these regions is not known. Furthermore, the equations admit discontinuous solutions, representing flows with shocks. Although both compression and expansion shocks are admitted by the equations, the expansion shocks are physically meaningless and must be excluded by the calculation method. A recent summary of the mathematical features of transonic flow is given by Nieuwland and Spee (1). Figure 1 shows typical transonic flow problems.

Linear problems of mixed elliptic-hyperbolic type have been treated by finite differences as well as finite elements. Of particular concern in these studies is the rate of convergence of the solution in the hyperbolic domain. For example, Filippov (2) solved the Tricomi equation by using centered differences in the elliptic region and a special backward difference scheme with a convergence rate of $O(h^{2/3})$ in the hyperbolic region (the nodes being the intersection of the characteristics known beforehand). Chu (3) treated the same problem using a Friedrich type-insensitive, finite difference approximation of a first-order symmetric positive system. Subsequently, Katsanis (4) showed that a similar finite difference scheme is pointwise divergent and has accuracy $O(h^{1/2})$ in L_2. Lesaint (5) has recently considered a Galerkin finite element approximation and has shown that if a triangulation of the domain in the simplexes with diameters less than h, with polynomial approximation of less than $k-1$ on each simplex is used, the approximate solutions converge at the rate of $O(h^{k-1})$ as $h \rightarrow 0$. Aziz and Leventhal (6) have studied the rate of convergence of Lesaint's scheme and have speculated that the rate of convergence may be $O(h^k)$, in contrast to the theoretical rate of convergence $O(h^{k-1})$. Their conclusion is based on computational results which indicate that with linear elements the rate of convergence is $O(h^2)$ instead of $O(h)$.

Fix and Gunzburger (7), however, has shown that Lesaint's results are sharp for higher order elements (in other words, there is at most a loss of one power of h for hyperbolic and mixed type systems). For linear elements only, the loss is not as great.

Transformations of the type used by Friedrichs (8) to a first-order symmetric, positive system have been derived for only a limited number of cases and apparently do not exist for the general case of interest. Consequently, Aziz, Fix and Leventhal (9) and Fix and Gunzburger (7) studied least-squares methods, which are independent of type; in other words, they may be used for elliptic, hyperbolic, or mixed first-order systems. Only under strong regularity assumptions, which apply primarily to elliptic systems, is convergence optimal in L_2. Under a weaker regularity assumption, convergence is not optimal in L_2, and there is, in fact, a loss in L_2 norm.

For linear problems with discontinuous solutions, mesh fitting has been used with finite differences, and discontinuous shape functions with finite elements.

The most successful treatment of shock-free, two-dimensional, transonic flow problems has been the method of complex characteristics introduced by Garabedian and Korn (10). They used a hodograph transformation to linearize the governing equations and introduced two additional (complex) independent variables. A hyperbolic problem is formulated in the complex domain, and finite difference methods are used to integrate the equations along characteristics. The real part of the solution is the physically meaningful result. Numerous shock-free airfoil shapes have been designed. An example, which was taken from Bauer, et al, (11) is shown in figure 2. Boerstoel and Uijlenhoet (12), on the other hand, used a different design procedure, which was based on the superposition of fundamental (singular) solutions in the hodograph plane (a generalization of Lighthill's work to lifting airfoils) and obtained comparable results (figure 3).

The <u>general</u> transonic problem, however, is basically nonlinear with the presence of embedded shock waves, and numerical methods are required to obtain solutions. A number of approaches have been suggested or are currently under study. The purpose of this article is

to briefly review these methods. Since space does not permit a complete comparison of many of the interesting results which have been reported, we refer the reader to the original papers.

We note here that each numerical approach must consider three separate topics. First, the discretization procedure must be selected. The two under consideration in this paper are finite difference and finite element. An important subtopic is the choice of the mesh description or finite element shapes. Usually irregular boundaries must be treated, and reasonably fine mesh spacing or elements are required in regions where the flow is varying rapidly (for example, stagnation points, and shocks). A poor mesh can cause loss of accuracy and slow convergence.

Second, the method of treating shock waves must be selected. Here, the two available approaches are (1) "shock-fitting" methods, where the shocks are explicitly treated as internal boundaries of unknown location and with applied jump relations, and (2) "shock-capturing" methods, where the embedded shock is smeared over several mesh points. For the latter approach, the difference equations must be in conservation form, and a positive artificial viscosity must be used to ensure that the proper weak solution is calculated. An intermediate approach is that of mesh-fitting advocated by MacCormack and Paullay (13), where the mesh changes with iteration to always coincide with the shock.

Third, a stable and convergent iteration procedure must be selected. Available approaches include the following: (1) embedding the steady problem in a higher dimension (unsteady) one that is more amenable to numerical integration (this method is referred to as the time-dependent approach); (2) generalized relaxation procedures, which may be viewed as a generalization of time-dependent methods; (3) semidirect methods, where a Poisson solver, for example, is used as one element of the iteration procedure; and (4) optimal control methods.

A proper combination of the above leads to an efficient computational method. Although we cannot treat in this article the interesting and important details of each approach, we hope that the reader will come to appreciate the extent of the current work in this subject and the problems that need to be solved. For the reader who is unfamiliar

with transonic flow, we note that accurate and efficient computational methods are needed to predict the performance of modern military and commercial aircraft, propulsion, and energy generation devices.

12.2 Governing Equations

For convenience, we list here the various forms of equations governing two-dimensional unsteady transonic flow and the jump conditions admitted by their weak solution. A reference for the derivation of the equations is Liepman and Roshko (14).

Euler Equation

$$W_t + F_x + G_y = 0 \text{ , where} \tag{12.1}$$

$$W = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho(\frac{1}{2}q^2 + e) \end{pmatrix}, \quad F = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho u(\frac{1}{2}q^2 + h) \end{pmatrix}, \quad G = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho v(\frac{1}{2}q^2 + h) \end{pmatrix} \tag{12.2}$$

and

$$q^2 = u^2 + v^2 \ , \quad e = C_v T \ , \quad h = C_p T \ , \quad p = \rho RT \quad . \tag{12.3}$$

For isentropic flows, the assumption $p \propto \rho^\gamma$ is used to eliminate the last equation of (12.1).

The weak solution admits the Rankine-Hugoniot jump conditions

$$[\![W]\!] \; S_t = - \; [\![F]\!] \; S_x - \; [\![G]\!] \; S_y \quad , \tag{12.4}$$

where the shock is described by the implicit relation $S(x,y,t) = 0$ and the symbol $[\![a]\!]$ represents the jump in the quantity a across the shock; in other words, $a^+ - a^-$.

Full Potential Equation

Conservative Form: $$-\rho_t = (\rho u)_x + (\rho v)_y \quad , \tag{12.5}$$

where
$$u = \phi_x + \cos\alpha \quad ,$$
$$v = \phi_y + \sin\alpha \quad ,$$
$$\frac{\rho}{\rho_\infty} = \left[\frac{1}{M_\infty^2} - \frac{\gamma-1}{2}\,(2\phi_t + u^2 + v^2 - 1)\right]^{\frac{1}{\gamma-1}} \quad , \tag{12.6}$$

and α is the angle of attack of the incoming flow of Mach number M_∞ .

Non-Conservative Form:

$$\phi_{tt} + 2u\phi_{xt} + 2v\phi_{yt} = (a^2 - u^2)\phi_{xx} - 2uv\phi_{xy} + (a^2 - v^2)\phi_{yy} \quad , \tag{12.7}$$

where
$$a^2 = \frac{1}{M_\infty^2} - \frac{\gamma-1}{2}\,(2\phi_t + u^2 + v^2 - 1) \tag{12.8}$$

$$\phi_{tx} = \phi_{xt} \quad , \quad \phi_{ty} = \phi_{yt} \quad . \tag{12.9}$$

Non-Conservative Form in Natural Coordinates:

$$\phi_{tt} + 2q\phi_{st} = (a^2 - q^2)\phi_{ss} + a^2\phi_{nn} \quad , \tag{12.10}$$

where
$$\phi_{ss} = \frac{1}{q^2}\,(u^2\phi_{xx} + 2uv\phi_{xy} + v^2\phi_{yy}) \tag{12.11}$$

$$\phi_{nn} = \frac{1}{q^2}\,(v^2\phi_{xx} - 2uv\phi_{xy} + u^2\phi_{yy}) \tag{12.12}$$

$$\phi_{st} = \left(\frac{u}{q}\right)\phi_{xt} + \left(\frac{v}{q}\right)\phi_{yt} \quad . \tag{12.13}$$

The jump conditions are

$$\rho\ [\![S_t]\!] = -\ [\![\rho u]\!]\ S_x - [\![\rho v]\!]\ S_y \tag{12.14}$$

$$S_t : S_x : S_y = [\![\phi_t]\!] : [\![\phi_x]\!] : [\![\phi_y]\!] \quad . \tag{12.15}$$

(12.16)

We note that equations (12.7) and (12.10) are in non-conservative form. Equations (12.15) and (12.16) must be added to eq. (12.7) or (12.10) to complete the formulation of the governing equations.

Transonic Small-Disturbance Equation

Conservative Form:

$$(M_\infty^2 \phi_t + 2M_\infty^2 \phi_x)_t = \left[\left(1 - M_\infty^2\right)\phi_x - \frac{\gamma + 1}{2} M_\infty^2 \phi_x^2 \right]_x + \left(\phi_y\right)_y \qquad (12.17)$$

Non-Conservative Form:

$$M_\infty^2 \phi_{tt} + 2M_\infty^2 \phi_{xt} = \left[\left(1 - M_\infty^2\right) - (\gamma + 1) M_\infty^2 \phi_x \right] \phi_{xx} + \phi_{yy} \qquad (12.18)$$

The jump conditions are

$$(M_\infty^2 [\![\phi_t]\!] + 2M_\infty^2 [\![\phi_x]\!]) S_t = \Big\{ (1 - M_\infty^2) [\![\phi_x]\!] - (\gamma + 1) M_\infty^2 \langle \phi_x \rangle [\![\phi_x]\!] \Big\} S_x + [\![\phi_y]\!] S_y , \qquad (12.19)$$

where the symbol $\langle a \rangle$ represents the average in the quantity a across the shock; in other words, $(a^+ + a^-)/2$.

Hodograph Equations

Small-Disturbance Case (Tricomi):

$$\left[(1 - M_\infty^2) - (\gamma + 1) M_\infty^2 u \right] \frac{\partial y}{\partial v} + \frac{\partial x}{\partial u} = 0 ,$$

$$\frac{\partial x}{\partial v} - \frac{\partial y}{\partial u} = 0 . \qquad (12.20)$$

Or, let $\quad y = J_v , \; x = J_u \; ; \qquad (12.21)$

hence $\quad \left[(1 - M_\infty^2) - (\gamma + 1) M_\infty^2 u \right] J_{vv} + J_{uu} = 0 . \qquad (12.22)$

Full-Potential Case (Chaplygin):

$$(a^2 - u^2)\ \frac{\partial y}{\partial v} + uv\left(\frac{\partial x}{\partial v} + \frac{\partial v}{\partial u}\right) + \left(a^2 - v^2\right)\ \frac{\partial x}{\partial u} = 0 \qquad (12.23)$$

$$\frac{\partial x}{\partial v} - \frac{\partial y}{\partial u} = 0 \quad .$$

or $$(a^2 - u^2)J_{vv} + 2uvJ_{uv} + (a^2 - u^2)\ J_{uu} = 0 \quad . \qquad (12.24)$$

12.3 Finite Difference Methods

Time-Dependent Methods

Magnus and Yoshihara (15) solved the unsteady Euler equations (12.1) using a Lax-Wendroff-type finite difference method with additional artificial viscosity. The system of equations is always hyperbolic in time and may be integrated as an initial value problem by using an explicit method. Magnus and Yoshihara assumed that $p \propto \rho^{\gamma}$ and calculated isentropic flow. For small-disturbance calculations, they used the following hyperbolic equations:

$$\begin{aligned} u_t &= \left[\left(1 - M_{\infty}^2\right) - (\gamma + 1)\, M_{\infty}^2 u\right] u_x + v_y \\ v_t &= u_y - v_x \quad , \end{aligned} \qquad (12.25)$$

and for full-potential (non-conservative calculation):

$$\begin{aligned} u_t &= (a^2 - u^2)u_x - 2uv + (a^2 - u^2)v_y \\ v_t &= u_y - v_x \ . \end{aligned} \qquad (12.26)$$

Convergence of these methods to the steady-state limit is very slow because of the limitation at the time step required by the CFL condition $\Delta t \leq \Delta x/a$, where a is the local speed of propagation of a sound wave.

Relaxation Methods

Murman and Cole (16) and Murman (17) introduced a type-dependent finite difference scheme to solve the steady-state version of eq. (12.17). In their method, backward differences (first-order accurate) are used in the supersonic zone, and centered differences in subsonic regions. Special operators at the sonic line and at the shock are needed to guarantee a conservation form. The four operators introduce artificial viscosity to eliminate expansion shocks, and they also ensure that the discretization matrix is positive definite.

A successive line overrelaxation method (marching in the flow direction) is used to solve the difference equations that are written on a regular Cartesian mesh system. Garabedian and Korn (10) extended this method to the steady version of the full potential equation (12.7) in non-conservative form. They used a coordinate transformation to map the exterior of the airfoil to the interior of a circle. The method is not unconditionally stable, however, because the coordinate system is misaligned with the flow direction.

Jameson (18) resolved this problem by using the natural or streamline coordinate system and the steady form of eq. (12.10). He introduced a rotated difference scheme in which backward differences are used for all terms contributing to ϕ_{ss} and centered differences are used for terms contributing to ϕ_{nn}, where s is the direction of the flow and n is normal to it. Jameson considered the iteration sequence as an artificial time (generalized relaxation) and derived a time-dependent equation to analyze the convergence properties of the scheme.

The combined works of Garabedian, Korn, and Jameson together with a boundary layer correction added by Bauer have been published in a handbook (Bauer, et al. (11)), which has been used extensively throughout world. We know, however, that the fact that the equations are in non-conservative form means that the embedded shock waves are incorrectly calculated. The error that is equivalent to a source has the same effect as a weakening of the shock by the airfoil boundary layer. These two effects are unrelated, and it is fortuitous that the error provides better agreement with data. Non-conservative forms, however, should not be used where shocks are present because they are incorrect and, in general, will lead to an incorrect solution. Calculation of internal and unsteady flows using non-conservative methods is strictly incorrect.

Jameson (19) introduced a fully conservative scheme for the full potential equation. Jameson pointed out that backward differences in the supersonic region are equivalent to the centered differences plus an artificial viscosity term. He solves the centered difference formula for eq. (12.5) (in steady form) with an additional "viscosity term" written in conservative form. For the small-disturbance equation, this approach leads to Murman's four operators (elliptic, parabolic point, hyperbolic, and shock point operators).

To see this point let the transonic small-disturbance equation be written in the form

$$f_x + g_y = 0 \; , \tag{12.27}$$

where $$f = K\phi_x - \frac{\gamma+1}{2}\phi_x^2 \; , \; g = \phi_y \; ,$$

and let the central difference approximation be

$$P_{ij} + Q_{ij} = 0 \; , \tag{12.28}$$

where $$P_{ij} = \frac{f_{i+\frac{1}{2},j} - f_{i-\frac{1}{2},j}}{\Delta x} \; ,$$

$$Q_{ij} = \frac{g_{i,j+\frac{1}{2}} - g_{i,j-\frac{1}{2}}}{\Delta y} \; .$$

The fully conservative scheme is simply to add to the left-hand side of the central difference equation the following terms:

$$-\mu_{ij}P_{ij} + \mu_{i-1,j}P_{i-1,j} \tag{12.29}$$

where $$\mu_{ij} = \begin{cases} 1 \text{ at supersonic points} \; , \\ 0 \text{ at subsonic points} \; . \end{cases}$$

An example of the effect of conservation form (Murman (17)) is shown in figure 4.

Relaxation methods for type-dependent difference equations have proven to be versatile and efficient. They have therefore been rapidly extended to include supersonic free stream cases and increasingly complex geometries. Calculation times are approximately one order of magnitude less than those required by time-dependent methods, and procedures discussed in the next section have led to yet another order of magnitude reduction in computing time. The principle restriction to finite-difference methods is the requirement of establishing a suitable mesh system for complex body shapes. For this reason, effort has been directed towards finite volume and finite element methods.

Semidirect Methods

Semidirect methods approach the transonic equation as an elliptic problem and make use of fast-solver or direct-inversion algorithms for elliptic operators. Since the system is nonlinear and non-elliptic, the algorithms must be applied iteratively and with some corrections; hence, the name "semidirect."

Consider the Newton-Ralphson linearization

$$J(\phi^n)\delta\phi = -R(\phi^n) \ , \qquad (12.30)$$

where $\delta\phi = \phi^{n+1} - \phi^n$,

$$R(\phi^n) = (K - \phi_x^n)\phi_{xx}^n + \phi_{yy}^n \ .$$

$$J(\phi^n) = (K - \phi_x^n)\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \phi_{xx}^n \frac{\partial}{\partial x} \ .$$

As a special case, $J(\phi^n)$ may be approximated by

$$\left.\begin{array}{ll} J_1 = K\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} & \text{(reduces to Rayleigh-Janzen)} \\ J_2 = \alpha\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \beta\dfrac{\partial}{\partial x} & \text{(unsteady transonic equation;} \\ & \alpha = K - \phi_x^n \ , \quad \beta = -2\text{) .} \end{array}\right\} \qquad (12.31)$$

For the small-disturbance equation written as a system of first-order equations, Martin and Lomax (20) suggested an iterative procedure very similar to Rayleigh-Janzen J_1 with the residual $R(\phi^n)$ evaluated using Murman's operators. Jameson (19) showed that this procedure fails to converge for purely supersonic flows unless an additional de-symmetrization term is added to J_2 . Backward differences are used for $\beta\frac{\partial}{\partial x}$, and a stability condition on α and β in terms of K is obtained from Von Neuman analysis. Hafez, et al. (21) point out that the stability condition in the supersonic region is equivalent to requiring that J_2 is a hyperbolic operator. For small-disturbance equations, fast solvers can be used for J_2 . Martin (22) has successfully calculated flows with large supersonic zones with this iteration scheme. Note that this elliptic method (including the stabilization term) can be viewed as a partially implicit integration procedure of the unsteady equation; namely,

$$\frac{u^{n+1} - u^n}{\Delta t} = \omega K u_x^{n+1} + v_y^{n+1} - u^n u_x^n + (1 - \omega)K u_x^n \qquad (12.32)$$

$$u_y^{n+1} - v_x^{n+1} = 0 \ ,$$

where, at each n step, a fast solver is used.

For the full potential equation, the de-symmetrization terms needed for convergence destroy the regular structure of the matrix needed in fast solvers. Jameson (19) introduced a hybrid method combining a fast solver and a relaxation procedure. The first step of the calculation is to rewrite the potential equation as a Poisson equation with the nonlinear terms on the right-hand side. An odd-even reduction algorithm is used to invert the matrix. The solution in the supersonic zone is then incorrect, and continued application of the fast solver will lead to divergence. The calculations, however, may be stabilized by executing one or more relaxation steps to correct the supersonic zone. Then, the calculations converge rapidly.

Related Studies

Other studies have concentrated on reducing the number of iterations required for convergence. Hafez and Cheng (23) and Caughey and Jameson (24) used extrapolation methods of the Aitken-Shanks type. For some cases these methods work well, while for other cases no substantial improvement is noted. ADI methods and approximate factorization schemes have recently been reported by Ballhaus, et al. (25). This approach appears quite promising. A multi-grid method reported by South and Brandt (26) works well for uniform mesh spacing, but deteriorates for the nonuniform meshes used in most practical calculations.

The above finite difference methods all use "shock-capturing" procedures. "Shock-fitting" procedures have been developed for the Euler equations by Moretti (27) in a series of papers. Application to transonic flow problems has been limited because of the slow convergence of time-dependent methods. Hafez and Cheng (28) and Yu and Seebass (29) have applied "shock-fitting" methods to the transonic small-disturbance equation for several model problems. Coarse meshes may be used with more accurate shock jumps and with reduced computing times. Complications arise, however, when intersecting shocks are present.

12.4 Finite Volumes

A major difficulty in finite difference methods is the application of boundary conditions when the boundaries are geometrically complex. For a Cartesian mesh, special finite difference schemes are needed near the boundary. The alternative approach of mapping the computational domain to a simple region is sometimes a formidable task (for example, realistic three-dimensional bodies). Finite volume methods use general nonorthogonal coordinates and consider the governing integral equations as balances of mass, momentum, and energy fluxes for each finite volume defined by the intersection of the coordinate surfaces.

Following McCormack, Rizzi (30) applied the finite volume method for the Euler equation of transonic flows and calculated the time-accurate convergence to a steady state. Factored explicit and implicit difference operators were used to accelerate the calculations (see figure 5).

Jameson and Caughey (31) have recently applied the finite volume method to the steady full-potential equation by using mixed-type flux operators. They noticed that the flux balance equation can be written in computational coordinates X,Y as

$$\frac{\partial}{\partial X}(\rho f) + \frac{\partial}{\partial Y}(\rho g) = 0 \quad , \tag{12.33}$$

where
$$\begin{pmatrix} f \\ g \end{pmatrix} = DJ^{-1}\begin{pmatrix} u \\ v \end{pmatrix} = D(J^T J)^{-1}\begin{pmatrix} \phi_X \\ \phi_Y \end{pmatrix} ,$$

where J the Jacobian of the tranformation

$$J = \begin{pmatrix} \dfrac{\partial x}{\partial X} & \dfrac{\partial x}{\partial Y} \\ \\ \dfrac{\partial y}{\partial X} & \dfrac{\partial y}{\partial Y} \end{pmatrix} \tag{12.34}$$

and $D = \det J$.

Instead of assuming a single mapping of the entire domain, Jameson and Caughey regard each rectangular cell in the computational domain as being separately mapped by a local bilinear mapping to an arbitrary quadrilateral cell in the physical domain.

Assuming a similar bilinear form for the potential, Jameson and Caughey evaluate the Jacobian and the derivatives of ϕ at the center of the cell. The flux balance leads to a <u>finite difference</u> formula, which corresponds closely to a finite element scheme based on the Bateman variational principle that the integral of the pressure $I = \int p \, dxdy$ is stationary. In this interpretation, the cells should be regarded as isoparametric elements. So far, the scheme is suitable only for subsonic flows. For transonic flows, an effective switch to an upwind difference scheme in the supersonic region is accomplished by adding on artificial viscosity. Thus, the flux balance equation is modified to become

$$\frac{\partial}{\partial X} (\rho f + P) + \frac{\partial}{\partial Y} (\rho g + Q) = 0 \quad , \tag{12.35}$$

where
$$P = -\mu f \nabla_X \rho \quad , \quad Q = -\mu g \Delta_Y \rho \quad ,$$

$$\mu = \frac{1}{D} \max \left[1, \quad 1 - \left(\frac{a^2}{q^2} \right) \right] \; ,$$

and Δ_X and Δ_Y denote upwind difference operators in the local X,Y directions. The resulting difference equations are solved by a generalized relaxation scheme. The method combines the advantage of finite elements for handling complicated geometries and the advantage of using simple finite difference schemes in the transformed coordinates. A typical calculation is shown in figure 6.

12.5 Finite Elements

Subsonic flows around airfoils and wings (elliptic boundary value problem) are calculated efficiently by finite elements (see, for example, Argyris and Mareczek (32) and Periaux (33)). For transonic flows, however, the governing potential equation is not elliptic, and the problem is complicated by the presence of embedded supersonic regions terminated by shock waves (or embedded subsonic regions in supersonic flows), as mentioned earlier. The application of finite elements to non-elliptic problems is currently an active field of research. In the following we discuss the development of finite element methods for transonic flows.

Hyperbolic Methods

Finite elements in space and finite differences in time can be applied directly to hyperbolic systems.* In particular, finite element Lax-Wendroff schemes were introduced by Wellford and Oden (34) and were used by Kawahara (35) for shallow-water equations (identical to transonic flow equations if γ is taken to be 2). For simplicity, consider the one-dimensional equation

$$\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} = 0 \quad . \tag{12.36}$$

The development of Lax-Wendroff scheme starts with the explicit relation:

$$u(t + \Delta t) = u(t) - \Delta t \left(\frac{\partial F}{\partial x}\right) + \frac{\Delta t^2}{2} \left[\frac{\partial}{\partial x}\left(A \frac{\partial F}{\partial x}\right)\right] , \tag{12.37}$$

where $A = A(u)$ is the Jacobian of $F(u)$.

A Galerkin finite element scheme is readily obtained by multiplying the equation by the shape functions and integrating in space, reducing the problem to a system of ordinary differential equations (in time). Such calculations based on "explicit" Lax-Wendroff schemes are usually slow (the finite element version can be made explicit by using the lumped mass formulation where the mass matrix is replaced by unity). A comparison between finite element and finite difference calculations for the shallow-water equation has been made by Kawahara (35). The results show the advantage of using finite elements in handling complicated geometry problems.

---

*See, for example, Oden and Reddy (49).

Implicit methods are generally chosen for their numerical stability; therefore, they are most often used when the stability bound of an explicit method (CFL condition) would be more restrictive than the desired accuracy bounds, as in transonic aerodynamics. Here we will consider a simple, straightforward application of an implicit Galerkin finite element procedure to the transonic small-disturbance equation with an additional artificial viscosity term (Wahlbin (36) studied the effect of additional artificial viscosity on finite element approximation of hyperbolic systems) by using the same system suggested by Magnus and Yoshihara (15):

$$u_t = K_\ell u_x + v_y + \varepsilon u_{xx} \quad ; \quad K_\ell = K_1 - K_2 u \tag{12.38}$$

$$v_t = u_y - v_x \quad , \tag{12.39}$$

where

$$K_2 = (\gamma + 1)M_\infty^2 \quad ,$$

$$K_1 = (1 - M_\infty^2) \quad ,$$

and εu_{xx} is the artificial viscosity term.

If we now represent the velocity in terms of a shape function ψ as

$$u = \psi_i(x,y)\ U_i(t) \tag{12.40}$$

$$v = \psi_i(x,y)\ V_i(t) \tag{12.41}$$

and apply a Galerkin procedure, eqs. (12.38) and (12.39) become

$$\iint \left(\psi_i\ U_{i,t}\ \psi_j\right) dx\, dy = \iint \left(K_\ell\ \psi_{i,x}\ U_i\ \psi_j\right) dx\, dy + \iint \left(\psi_{i,y}\ V_i\ \psi_j\right) dx\, dy + \iint \left(\varepsilon\ \psi_{i,xx}\ U_i\ \psi_j\right) dx\, dy \tag{12.42}$$

and

$$\iint \left(\psi_i\ V_{i,t}\ \psi_j\right) dx\, dy = \iint \left(\psi_{i,y}\ U_i\ \psi_j\right) dx\, dy - \iint \left(\psi_{i,x}\ V_i\ \psi_j\right) dx\, dy \quad . \tag{12.43}$$

These equations can be written in the more compact form (a system of ordinary differential equations):

$$M_{ij} \frac{dU_i}{dt} = K_\ell K_{ji} U_i + C_{ji} V_i + \varepsilon D_{ij} U_i \tag{12.44}$$

$$M_{ij} \frac{dV_i}{dt} = C_{ji} U_i - K_{ji} V_i \tag{12.45}$$

by defining the matrices

$$M_{ij} = \iint \psi_i \psi_j \, dx\, dy \ , \tag{12.46}$$

$$K_{ji} = \iint \frac{\partial \psi_i}{\partial x} \psi_j \, dx\, dy \ , \tag{12.47}$$

$$C_{ji} = \iint \frac{\partial \psi_i}{\partial y} \psi_j \, dx\, dy \ , \tag{12.48}$$

$$D_{ji} = \iint \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} \, dx\, dy \ . \tag{12.49}$$

For the special case of bilinear shape functions on rectangular elements, the functions ψ_i become

$$\psi_1 = \left(1 - \frac{x}{h_1}\right) \left(1 - \frac{y}{h_2}\right) \ , \quad \psi_2 = \frac{x}{h_1} \left(1 - \frac{y}{h_2}\right) \ , \tag{12.50}$$

$$\psi_3 = \frac{x}{h_1} \frac{y}{h_2} \ , \quad \psi_4 = \left(1 - \frac{x}{h_1}\right) \frac{y}{h_2} \ , \tag{12.51}$$

so that the matrices C , D , K , and M become

$$C_{ji} = \frac{h_1}{12} \begin{pmatrix} -2 & -1 & 1 & 1 \\ -1 & -2 & 2 & 1 \\ -1 & -2 & 2 & 1 \\ -2 & -1 & 1 & 2 \end{pmatrix} \tag{12.52}$$

$$D_{ji} = \frac{h_2}{6h_1} \begin{pmatrix} 2 & -2 & -1 & 1 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & -2 \\ 1 & -1 & -2 & 2 \end{pmatrix} \tag{12.53}$$

$$K_{ji} = \frac{h_2}{12} \begin{pmatrix} -2 & 2 & 1 & -1 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & -2 \\ -1 & 1 & 2 & -2 \end{pmatrix} \tag{12.54}$$

and

$$M_{ji} = \frac{h_1 h_2}{36} \begin{pmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{pmatrix} \tag{12.55}$$

Alternatively, Wellford and Hafez (37) suggested the following algorithm: First, the iterative procedure is set up (in other words, the time-dependent terms are discretized); namely

$$\frac{\alpha}{\Delta t}(u^{n+1} - u^n) = (K_1 u^{n+\frac{1}{2}})_x - K_2/2(u^n u^{n+\frac{1}{2}})_x + v_y^{n+\frac{1}{2}} + \gamma u_{xx}^{n+\frac{1}{2}} - \delta u^{n+\frac{1}{2}} \quad , \tag{12.56}$$

$$\frac{\beta}{\Delta t}(v^{n+1} - v^n) = u_y^{n+\frac{1}{2}} - v_x^{n+\frac{1}{2}} \quad , \tag{12.57}$$

where

$$u^{n+\frac{1}{2}} = \frac{u^{n+1} + u^n}{2} \quad .$$

Second, a Galerkin finite element is applied (in space). Stability and convergence of the algorithm is analyzed. Preliminary results are encouraging (see fig. 7). If β is chosen to be zero or $O(\Delta t)^2$, the unsteady equation is retained.

In passing, we mention that instead of the Galerkin method, the least-squares method may be used (in space) similar to Carasso's (38) treatment of the wave equation and a coupled system of wave and heat equations.

Modifications of the above procedure are under study (no results are available yet).

With the above procedure, eqs. (12.56) and (12.57) may be replaced by

$$\frac{\alpha}{\Delta t}\left(u^{n+1}-u^{n}\right) = K_1\left(u^{n+\frac{1}{2}}\right)_x - K_2\left(u^n u^{n+\frac{1}{2}}\right)_x + \left(v_y\right)^{n+\frac{1}{2}}$$
$$+ \left\{\mu\left[K\left(u^{n+\frac{1}{2}}\right)_x - \left(u^n u^{n+\frac{1}{2}}\right)_x\right]\right\}_x$$
$$+ K_2/2\left(u^n\right)^2_x - \delta u^{n+\frac{1}{2}} \quad , \tag{12.58}$$

$$\frac{\beta}{\Delta t}\left(v^{n+1}-v^{n}\right) = u_y^{n+\frac{1}{2}} - v_x^{n+\frac{1}{2}} \quad , \tag{12.59}$$

where μ is a switching operator* (its value is 1 in supersonic regions and zero in subsonic regions). Note that mass is conserved globally in space and time. Note also that equations (12.58) and (12.59) describe a flow that is irrotational only at the steady-state limit. There is some evidence supporting the belief that convergence will be accelerated if irrotationality is preserved during iteration, as we discuss later.

For this purpose let

$$u = \phi_x \ , \quad v = \phi_y \ , \quad w = \phi_t \tag{12.60}$$

Hence, the unsteady transonic small-disturbance equation (12.17) can be written in a form of a system of first-order equations:

$$v_t = \left(K_1 u - K_2/2u^2\right)_x + v_y - 2\alpha w_x - \delta w \tag{12.61}$$

$$u_t = w_x$$

$$v_t = w_y \quad ,$$

where from stability analysis, δ must vanish in the supersonic region. The system of equations (12.61) is always hyperbolic in time for both subsonic and supersonic flows as long as

$$\alpha^2 + K_1 - K_2 u > 0 \quad . \tag{12.62}$$

---

*For the concept of conservative switching see Jameson (19) and Beam and Warming (39).

An artificial viscosity term in conservative form may be added as discussed earlier. We mention that the introduction of the artificial viscosity term to the first-order system of equations is equivalent to backward-differencing (if the right coefficient is used as in the Lax-Wendroff scheme, for example).

The (Galerkin) finite element implementation of the linearized system is given by

$$\left.\begin{aligned} M_{i,j}\, W_{i,t} &= K_\ell\, K_{ji}\, U_i + C_{ji}\, V_i - \delta W_{ji} \\ M_{i,j}\, U_{i,t} &= K_{ji}\, W_i \\ M_{i,j}\, V_{i,t} &= C_{ji}\, W_i \quad . \end{aligned}\right\} \qquad (12.63)$$

For the full potential equation, the idea may be extended readily to

$$\begin{aligned} w_t &= (a^2 - u^2)u_x - uv(v_x + u_y) - (a^2 - v^2)\, v_y - 2uw_x - 2vw_y \\ u_t &= w_x \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (12.64) \\ v_t &= w_y \quad . \end{aligned}$$

The first equation of the above system can be written in a conservative form by multiplying by ρ/a^2 .

We note here that an approach in the opposite direction (stressing the dynamics rather than the kinematics of the flow) was adopted by Phares and Kneile (40). The velocity field is calculated from the momentum equations:

$$\frac{du}{dt} + \frac{1}{(\gamma - 1)M_\infty^2} \quad T_x = \varepsilon\nabla^2 u \qquad (12.65)$$

$$\frac{dv}{dt} + \frac{1}{(\gamma - 1)M_\infty^2} \quad T_y = \varepsilon\nabla^2 v \quad , \qquad (12.66)$$

where $\left(\frac{d}{dt} = \frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y}\right)$ and $\varepsilon > 0$.

Here, the temperature T is used rather than the density (to avoid calculating fractional powers) and is governed by the equation

$$\frac{dT}{dt} + (\gamma - 1)T(u_x + v_y) = 0 . \tag{12.67}$$

A (Galerkin) finite element implementation reduces the problem to a system of ordinary differential equations (in time). Direct numerical integration of the system gave unstable results. The integration procedure was modified to reset the variable T after a specific number of integration steps using (steady) Bernoulli's equation, as follows:

$$T = 1 - \left(\frac{\gamma - 1}{2}\right)M_\infty^2 \, (u^2 + v^2 - 1) . \tag{12.68}$$

Some results are shown in fig. (8). (When a shock occurs, twenty to fifty minutes on a IBM 370/16S are required. This method is almost 100 times slower than Jameson's (19) program!)

Mixed-Type Methods

In this section we discuss a finite element analog of backward differences as well as the use of artificial time and artificial viscosity.

Following Hafez, et al. (21) consider linear hat functions in y and Hermite cubics in x on rectangular elements. The cubic polynomial on $0 \le x \le \Delta x$, which takes on the four prescribed values ϕ_0, ϕ_{x0}, ϕ_1, and ϕ_{x1} is

$$\left|X_i\right| = \left(H_{00} , H_{10} , H_{01} , H_{11}\right) \begin{pmatrix} \left|\phi_0\right| \\ \left|\phi_{x_0}\right| \\ \left|\phi_1\right| \\ \left|\phi_{x_1}\right| \end{pmatrix} \tag{12.69}$$

with
$$H_{00} = 2\theta^3 - 3\theta^2 + 1$$

$$H_{01} = -2\theta^3 + 3\theta^2$$

$$H_{10} = (\theta^3 - 2\theta^2 + \theta)\Delta x$$

$$H_{11} = (\theta^3 - \theta^2)\Delta x$$

$$\left(\theta = \frac{x}{\Delta x}\right).$$

In the subsonic region, the contribution of the neighboring element will be included through the assembly of the elemental expression into the global system. In the supersonic region, the stationary value with respect to $\phi_x(0)$ and $\phi_x(\Delta x)$, assuming $\phi(0)$ and $\phi(\Delta x)$ are known, will give two algebraic equations that will be used to solve for $\phi(\Delta x)$ and $\phi_x(\Delta x)$; namely,

$$\int_0^{\Delta x} \begin{pmatrix} H_{10} \\ H_{11} \end{pmatrix} \left[\frac{\partial}{\partial x}\left(M \frac{\partial}{\partial x}\right) + K\right] \left|X\right| \, dx = 0 \tag{12.70}$$

or
$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \left|\phi_1\right| \\ \left|\phi x_1\right| \end{pmatrix} = - \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} \left|\phi_0\right| \\ \left|\phi_{x_0}\right| \end{pmatrix} + \begin{pmatrix} \left|f_0\right| \\ \left|f_1\right| \end{pmatrix} . \tag{12.71}$$

Note, no upstream effect is allowed in the supersonic region. It has been shown, however, that these schemes are not unconditionally stable. In general, extra artificial viscosity as well as artificial inertia (regularization) terms are needed. This subject is discussed by Hafez, et al. (21), who, using rectangular elements and line relaxation methods, calculated flows over a parabolic arc airfoil. The results are shown in fig. (9). Accuracy and convergence (of the iteration) are comparable to Murman's finite difference calculations.

An alternative procedure is to zero out the upstream effect in the supersonic zone. (In the process of assembling the element matrices

into the system matrix, the rows corresponding to the nodes along the upstream side of any element in the supersonic zone are zeroed out.) In this case, we obtain

$$\int_0^{\Delta x} \begin{pmatrix} H_{01} \\ H_{11} \end{pmatrix} \left[\frac{\partial}{\partial x}\left(M \frac{\partial}{\partial x}\right) + K\right] \left\{X\right\} \quad dx = 0 \tag{12.72}$$

or

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} \left|\phi_1\right| \\ \left|\phi_{x_1}\right| \end{pmatrix} = - \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} \begin{pmatrix} \left|\phi_0\right| \\ \left|\phi_{x_0}\right| \end{pmatrix} + \begin{pmatrix} \left|f_0\right| \\ \left|f_1\right| \end{pmatrix} . \tag{12.73}$$

A similar procedure was used by Chan and Brashears (41) with a least-squares method. Their work will be discussed later.

Finally, we mention here the work of Fix (42). Fix constructed a mixed variational formulation using ϕ , u , v variables in the elliptic region and u and v only in the hyperbolic region. Matching along the sonic line was demonstrated analytically, and some results for the linear Tricomi equation were given.

Extensions of these methods to the full potential equation are not trivial. Unlike the small-disturbance case, the streamlines are not almost parallel to the x axis. The flow direction is completely unknown. Here, the difficulty arises in the application of finite element methods (in the space as well as time-like direction) to the supersonic zone* and the change of strategy across the sonic line and the shock wave.

*The only work available in literature for finite elements in space and time for hyperbolic equations is that of Lesaint (43).

Elliptic Methods

In this section we will discuss mixed variational principles and dual iterative procedures, optimal control methods, and least squares.

Mixed Variational Principles. Consider the functional

$$J(\phi) = \tfrac{1}{2} \int_{\Omega} (K_1\phi_x^2 + \phi_y^2)\, dx\, dy - \frac{1}{6} \int_{\Omega} K_2\phi_x^3\, dx\, dy \quad , \tag{12.74}$$

where $K_1 = 1 - M_\infty^2$, $K_2 = (\gamma + 1)M_\infty^2$.

The Euler equation corresponding to this functional (setting the first variation equal to zero) is the transonic small-disturbance equation. The second variation, however, for arbitary η satisfying the boundary conditions is

$$\delta^2 J = \int_{\Omega} (K_1 - K_2\phi_x)\eta_x^2 + \eta_y^2\, dx\, dy \quad , \tag{12.75}$$

which is positive definite only for subsonic flows. We obtain the same conclusion for the full potential equation if we start from the Bateman principle. A mixed formulation is given by Wellford and Hafez (44) in terms of ϕ and u .

$$J(\phi,u) = \tfrac{1}{2} \int_{\Omega} (K_1\phi_x^2 + \phi_y^2)\, dx\, dy - \frac{K_2}{2} \int_{\Omega} u^2\phi_x\, dx\, dy + \frac{K_2}{3} \int_{\Omega} u^3\, dx\, dy \quad . \tag{12.76}$$

The Euler equations are

$$\delta_\phi I = 0 \quad \rightarrow \quad K_1\phi_{xx} - K_2uu_x + \phi_{yy} = 0 \tag{12.77}$$

$$\delta_u J = 0 \quad \rightarrow \quad K_2u(u - \phi_x) = 0 \quad , \tag{12.78}$$

and the second variations are

$$\delta_\phi^2 J = \int_\Omega (K_1 \eta_x^2 + \eta_y^2)\ dx\ dy \tag{12.79}$$

$$\delta_u^2 J = \int_\Omega (2u - \phi_x)\eta^2\ dx\ dy \tag{12.80}$$

$$\delta_{u\phi}^2 J = - \int_\Omega \frac{K_2}{2}\ u(\eta^2)_x\ dx\ dy \quad , \tag{12.81}$$

where η is a variation in ϕ or u .

Note that $\delta_\phi^2 J \geq 0$, $\delta^2 J \lesseqgtr 0$ for $u \lesseqgtr 0$, and $\delta_{u\phi}^2 J = 0$ for $u = 0$ (when the first variations vanish).

The functional $J(\phi,u)$ can be modified to include a natural Neuman boundary condition $\phi_n = g(x)$ along a portion of the boundary $\partial\Omega_1$ by adding a line integral

$$- \int_{\partial\Omega_1} \frac{\partial g}{\partial x}\ \phi(s)\ ds \quad . \tag{12.82}$$

The solution to our problem can be characterized as

$$\begin{aligned} \underset{u}{\text{Min}}\ \underset{\phi}{\text{Min}}\ I(\phi,u) &= I(\phi^*,u^*) \quad , \quad \phi_x^* > 0 \\ \underset{u}{\text{Max}}\ \underset{\phi}{\text{Min}}\ I(\phi,u) &= J(\phi^*,u^*) \quad , \quad \phi_x^* > 0 \quad . \end{aligned} \tag{12.83}$$

A finite element implementation is obtained as follows: Divide the domain into elements. On each element, ϕ and u are approximated by

$$\phi_e = \Psi_i(c,y)\phi_i \tag{12.84}$$

$$u_e = \beta_i(x,y)u_i \quad . \tag{12.85}$$

Hence,

$$J(\phi_e,u_e) = \frac{1}{2} K_{ij}\phi_i\phi_j - \frac{1}{2} L_{ijK}U_iU_j\phi_K + \frac{1}{3} M_{ijK}U_iU_jU_K - F_j\phi_j \quad , \tag{12.86}$$

where

$$K_{ij} = \int_{\Omega_e} (K_1 \Psi_{i_x} \Psi_{j_x} + \Psi_{i_y} \Psi_{j_y}) \, dx \, dy \quad ,$$

$$L_{ijk} = \int_{\Omega_e} (K_2 \beta_i \beta_j \Psi_{k_x}) \, dx \, dy \quad ,$$

$$Mijk = \int_{\Omega_e} (K_2 \beta_i \beta_j \beta_k) \, dx \, dy \quad ,$$

$$F_j = \int_{\partial\Omega_1} \frac{dg}{dx} \Psi_j \, ds \; .$$

Setting

$$\frac{\partial J(\phi_e, u_e)}{\partial \phi_j} = 0 \tag{12.87}$$

$$\frac{\partial J(\phi_e, u_e)}{\partial U_i} = 0 \tag{12.88}$$

leads to

$$\begin{aligned} &K_{ij}\phi_i - \tfrac{1}{2} L_{iKj} U_i U_K - F_j = 0 \\ &M_{ijK} U_j U_K - L_{ijK} U_j \phi_K = 0 \quad . \end{aligned} \tag{12.89}$$

The corresponding equations for the entire domain are obtained with standard assembly techniques.

These systems of equations may be solved simultaneously. Alternatively, Wellford and Hafez (44) introduced a dual iterative method in which a direct Poisson's solver is used for ϕ and a gradient method, which locally accounts for the sign change of $\delta_u^2 I$ for U . An artificial viscosity term was needed (otherwise the procedure diverges).

The system of equations (12.89) is solved as follows:

$$\left.\begin{aligned} &K_{ij}\phi_i^{n+1} = \tfrac{1}{2} L_{iKj} U_i^n U_K^{n+1} + F_j \quad , \\ &U_j(n+1) = U_j^{(n)} + C \frac{\partial J^{(n)}}{\partial U_j} - \varepsilon_1 D_{ij} U_i^{(n)} \\ &\qquad - \varepsilon_2 E_{ij} U_i^{(n)} \quad , \end{aligned}\right\} \tag{12.90}$$

where

$$D_{ij} = \iint \beta_{ix}\beta_j \, dx\, dy \quad ,$$

$$E_{ij} = \iint \beta_{ix}\beta_{jx} \, dx\, dy \quad ,$$

$$C > 0 \qquad \text{for} \qquad \delta^2_{u_j} J^{(n)} \geq 0 \quad ,$$

$$C < 0 \qquad \text{for} \qquad \delta^2_{u_j} J^{(n)} > 0 \quad .$$

Figure 10 shows the results of Wellford and Hafez (44). Hafez, et al. (21) discussed the extension to the full potential equation in terms of ϕ , u , and v :

$$J(\phi,u,v) = \frac{\lambda}{2}\iint \left\{ \left[(\phi_x - u)^2 + (\phi_y - v)^2 \right] + \rho \left[(u^2 - u\phi_x) + (v^2 - v\phi_y) \right] - C\rho^\gamma \right\} dx\, dy \quad , \tag{12.91}$$

where $C = 1/\gamma$.

$$\frac{\partial J}{\partial \phi} = 0 \quad \rightarrow \quad \lambda(\phi_{xx} + \phi_{yy}) = \left[(\rho u)_x + (\rho u)_y \right] + \lambda(u_x + v_y) \quad , \tag{12.92}$$

$$\frac{\partial I}{\partial u} = 0 \quad \rightarrow \quad \lambda(u - \phi_x) = \rho(\phi_x - u) + \rho_u(u\phi_x - u^2) + \rho_v(v\phi_y - v^2) \quad , \tag{12.93}$$

$$\frac{\partial I}{\partial v} = 0 \quad \rightarrow \quad \lambda(u - \phi_y) = \rho(\phi_y - v) + \rho_u(u\phi_x - u^2) + \rho_v(v\phi_y - v^2) \quad . \tag{12.94}$$

λ is a free parameter.*

Additional regularization terms are needed as in the transonic small-disturbance case. (Note that setting $u = \phi_x$ and $v = \phi_y$ leads to the Rayleigh-Janzen procedure.)

* There is no λ that renders second-order variations of J always positive definite. Jameson (45) independently came to the same conclusion for similar functions.

Optimal Control Methods

Following Glowinski and Pironneau (46), we consider the following minimization problem:

$$\int\left[(\phi_x - u)^2 + (\phi_y - v)^2\right] \; dx\, dy \tag{12.95}$$

with the constraints

$$\phi_{xx} + \phi_{yy} = \omega_1 \left[(K - u)u_x + v_y\right] + (u_x + v_y) \quad . \tag{12.96}$$

Using gradient method, we obtain

$$\left.\begin{aligned} \delta u &= -\omega_2(u - \phi_x) \\ \delta v &= -\omega_3(v - \phi_y) \quad . \end{aligned}\right\} \tag{12.97}$$

For the full potential equation, the procedure is the same, except the constraint equation becomes

$$\phi_{xx} + \phi_{yy} = \omega_1 \left[(\rho u)_x + (\rho u)_y\right] + (u_x + v_y) \quad , \tag{12.98}$$

where $$\rho = \left[1 + \frac{\gamma - 1}{2} \; (u^2 + v^2)\right]^{\frac{1}{\gamma - 1}} \quad .$$

For transonic flow calculations, expansion shocks must be excluded. Recently Glowinski and Pironneau (46) discussed methods to impose the entropy inequality*:

$$\Delta\phi = \phi_{xx} + \phi_{yy} \leq M < \infty \quad . \tag{12.99}$$

---

* The entropy inequality is automatically imposed if a positive artificial viscosity is added.

or in the weak forms

$$\int_\Omega \phi\Delta\phi \, dx \, dy \leq M \int_\Omega \phi \, dx \, dy \tag{12.100}$$

$$- \int_\Omega \nabla\phi\cdot\nabla\phi \, dx \, dy \leq M \int_\Omega \phi \, dx \, dy \, . \tag{12.101}$$

A penalty functional is added to the cost functional in the form

$$\varepsilon \int_\Omega \left[(\Delta\phi)^+\right]^2 dx \, dy \, , \quad \varepsilon > 0 \tag{12.102}$$

$$(\Delta\phi)^+ = \sup\,(0,\Delta\phi) \, .$$

Alternatively, the weak forms may be used in the following way:

Let $$\Delta\phi_o = M \, , \; \frac{\partial\phi_o}{\partial n} = g \quad . \tag{12.103}$$

The variational formulation is

$$\int_\Omega \nabla\phi_o\cdot\nabla\phi \, dx \, dy = - \int_\Omega M(x) \, \phi \, dx \, dy + \int_{\partial\Omega_1} g\phi \, ds \quad . \tag{12.104}$$

Hence, $$- \int_\Omega \nabla(\phi - \phi_o)\cdot\nabla\phi \, dx \leq 0 \tag{12.105}$$

$$\frac{\partial}{\partial n}(\phi - \phi_o) = 0 \quad . \tag{12.106}$$

The above linear inequality constraint may be relaxed by adding to the right-hand side a small nonnegative functional.

Finally, Glowinski and Pironneau (46) explicity imposed that the downstream normal velocity to the shock be less than the upstream one. (However, they did not impose the condition that flow upstream of the shock must be supersonic!) A combination of these formulations were tried and gave encouraging results, which are shown in figure 11.

Convergence of Iterative Solutions of Mixed Variational Principles and Optimal Control Methods

Modified steepest descent methods, (conjugate gradient methods etc.) were used by Glowinski, et al. (47), to accelerate the convergence of iterations.

More importantly, the choice of the control parameter (and the extra variables in the mixed variational principles) are crucial in this respect. For example, the mixed variational principle

$$I(\phi,\psi) = \frac{1}{2}\iint (K_1\phi_x^2 + \phi_y^2)\, dx\, dy - \frac{1}{2}\iint K_2\psi_x^2\phi_x\, dx\, dy + \frac{1}{3}\iint \psi_x^3\, dx\, dy \qquad (12.107)$$

with its Euler equations

$$\left.\begin{aligned} K_1\phi_{xx} + \phi_{yy} &= K_2\psi_x\psi_{xx} \\ K_2(\phi_x\psi_x - \psi_x^2)_x &= 0 \end{aligned}\right\} \qquad (12.108)$$

and the constraint minimization problem (Glowinski and Pironneau (46))

$$\operatorname*{Min}_{\psi} \iint_\Omega \rho^\alpha(\psi)\ |\nabla(\phi - \psi)|^2\, dx\, dy\ , \quad \alpha = 0 \text{ or } \alpha = 1 \qquad (12.109)$$

with the constraint $-\nabla\left[\rho(\psi)\nabla\phi\right] = 0$, depend, in these formulations, on the potential functions ψ. Hence, during iteration, irrotationality is preserved. This drastically affects the convergence (Glowinski, et al. (47) reports that the number of iterations are reduced by almost a factor of 100).

Least Squares

Aziz, Leventhal, and Fix (8), and Fix and Gunzburger (7) studied least squares applied to a system of first-order equations of general type, as follows:

$$\operatorname*{Min}_{u,v} \iint (K_\ell u_x + v_y)^2 + (u_y - v_x)^2\, dx\, dy \qquad (12.110)$$

and

$$\underset{\phi,u,v}{\text{Min}} \iint \left[(\phi_x - u)^2 + \lambda_1(\phi_y - v)^2 + \lambda_2(K_\ell u_x + v_y)^2\right] dx\, dy \quad . \tag{12.111}$$

Results were reported only for the Tricomi equation ($K_\ell = f(y)$).
For the second-order potential equation Chan and Brashears (41) used

$$\underset{\phi}{\text{Min}} \iint \left[(K - \phi_x)\phi_{xx} + \phi_{yy}\right]^2 dx\, dy \quad . \tag{12.112}$$

Unless the upstream effects are cancelled at supersonic nodes (through a special assembly procedure), calculations fail to converge. The validity of this procedure is questioned since the solution obtained (fig. 12) does not minimize the functional (12.112) anymore (because of the modification of the system matrix).

The functional (12.112), however, may be constrained in such a way that upstream effects are annihilated. For example, consider

$$\underset{\phi}{\text{Min}} \iint \left\{(K\phi_x - \tfrac{1}{2}\phi_x^2)_x + (\phi_y)_y + \left[\mu(K\phi_x - \tfrac{1}{2}\phi_x^2)_x\right]_x\right\}^2 dx\, dy \ , \tag{12.113}$$

where μ is a switching operator (μ is one in supersonic regions and zero in subsonic regions).

Shock-Fitting

Fost, Oden and Wellford (48) used discontinuous shape functions to fit a shock in a one-dimensional elastodynamic problem. The same procedure was applied to two-dimensional transonic flows by Hafez, et al. (21).

The procedure depends on the jump conditions admitted by the weak solution of the time-dependent equation in terms of shock speeds, as well as Hadamard kinematical relations of the propagation of singularity surfaces. Hafez and Murman (50) used a similar procedure for shock fitting in transonic type-dependent finite-difference relaxation calculations of full potential equation.

12.6 Conclusions

In this article we have reviewed the computational methods for inviscid transonic flows. Finite difference calculations using time-dependent methods, generalized relaxation methods, and elliptic methods have been discussed, and parallel developments of finite element analogs have been outlined. Although we believe that it is too early to conclude when the finite element approach will provide a competitive efficient method for calculating transonic flows, there is some evidence indicating that this is already true for some problems. Fair comparisons between finite element and finite difference methods have been made only for special cases. On one hand, for linear symmetric positive operators, the works of Lesaint (5,43) and Katsanis (4) show the theoretical advantage of using finite elements. Also, the work of Kawahara (35), in which Lax-Wendroff finite differences and finite elements were used for shallow-water equations, shows that the extra complication involved with the finite element procedures pays off; in particular, for problems of geometrical complexity. On the other hand, for transonic small-disturbance equations using linear shape functions on triangle (rectangulars) and line relaxation methods, Hafez, et al. (21), showed that finite elements are comparable to Murman's finite difference calculations. The applicability of this conclusion to the full potential calculations remains to be seen. The only available results are those of Glowinski and Pironneau (46), which seem to be promising, but not completely satisfactory. Perhaps the state of the art today is represented by Jameson's type-dependent finite difference relaxation calculations. For three-dimensional, realistic bodies, Jameson and Caughey (31) has recently introduced the "finite volume method," which is actually a combination of finite elements and finite differences. Future work should provide a good comparison between the advantages and disadvantages of using finite differences, finite volumes, finite elements, and maybe other techniques, for realistic, three-dimensional calculations.

References

(1) G. Y. Nieuwland and B. M. Spee, "Transonic Airfoils: Recent Developments in Theory, Experiment, and Design," Annual Review of Fluid Mechanics, V. 5, Annual Reviews, Inc., California (1973).

(2) A. Filippov, "On Difference Methods for the Solution of the Tricomi Problem," Izv. Akad. Nauk SSR. Ser. Mat., V. 21, pp. 73-88 (1957).

(3) C. K. Chu, "Type-Insensitive Difference Schemes," Ph.D. Thesis, New York University (1959).

(4) T. Katsanis, "Numerical Solution of Tricomi Equation Using Theory of Symmetric Positive Differential Equations," SIAM J. Numer. Anal., no. 2, pp. 236-253 (1969).

(5) P. Lesaint, "Finite Element Methods for Symmetric Hyperbolic Equations," Numer. Math., V. 21, pp. 244-255 (1973).

(6) A. K. Aziz and S. Leventhal, "Numerical Solution of Partial Differential Equations III," Ed. B. Hubbard, Academic Press, New York (1976).

(7) J. Fix and M. D. Gunzburger, "On Least Squares Approximations to Indefinite Problems of the Mixed Type," I Case Report No. 76-26, August, 1976.

(8) K. O. Friedrichs, "Symmetric Positive Linear Differential Equations," Comm. Pure Appl. Math., V. 11, pp. 333-418 (1958).

(9) A. K. Aziz, G. Fix, and S. Leventhal, "On Optimal Error Estimates for Finite Element Approximation for First Order Systems, to appear in Numer. Math..

(10) P. R. Garabedian and D. G. Korn, "Analysis of Transonic Airfoils," Comm. Pure Appl. Math., V. 24, pp. 841-851 (1972).

(11) F. Bauer, P. Garabedian, D. Korn, and A. Jameson, Supercritical Wing Sections II, Springer Verlag, New York (1975).

(12) J. W. Boerstoel and R. Uijlenhoet, "Lifting Aerofoils With Supercritical Shockless Flow," ICAS Paper No. 70-15 (1970).

(13) R. W. MacCormack and A. J. Paullay, "The Influence of the Computational Mesh On Accuracy for Initial Value Problems With Discontinuous or Nonunique Solutions," Computers and Fluids, V. 2, pp. 339-361 (1974).

(14) H. W. Liepman and A. Roshko, Elements of Gas Dynamics, John Wiley and Sons (1957).

(15) R. Magnus and H. Yoshihara, "Inviscid Transonic Flow Over Airfoils," AIAA Journal, V. 8, pp. 2157-2162 (1970).

(16) E. M. Murman and J. Cole, "Calculation of Plane Steady Transonic Flow," AIAA J., V. 9, no. 1, pp. 114-121 (1971).

(17) E. M. Murman, "Analysis of Embedded Shock Waves Calculated by Relaxation Methods," AIAA J., V. 12, no. 5, pp. 626-633 (1973).

(18) A. Jameson, "Iterative Solution of Transonic Flows Over Airfoils and Wings," Comm. Prob. Appl. Math, V. 2, pp. 283-309 (1974).

(19) A. Jameson, "Transonic Potential Flow Calculation Using Conservation Form," Proceedings of the AIAA Third Computational Fluid Dynamics Conference, Hartford, Connecticut, June 1975.

(20) E. D. Martin and H. Lomax, "Rapid Finite Difference Computation of Subsonic and Transonic Aerodynamic Flows," AIAA Paper 74-11 (1974).

(21) M. M. Hafez, L. C. Wellford, C. L. Merkle, and E. M. Murman, "Numerical Computation of Transonic Flows by Finite-Element and Finite-Difference Methods," Flow Research Report No. 70 (January 1977).

(22) E. D. Martin, "A Fast Semi-Direct Method for Computing Transonic Aerodynamic Flows," Proceedings of Second AIAA Computational Fluid Dynamics Conference, Hartford, Connecticut, pp. 162-174 (June 1975).

(23) M. M. Hafez and H. K. Cheng, "Shock Fitting Applied to Relaxation Solutions of Transonic Small Disturbance Equations," AIAA J., (June 1977).

(24) D. A. Caughey and A. Jameson, "Accelerated Iterative Calculation of Transonic Nacelle Flow Fields," AIAA Paper 76-100 (1976).

(25) W. F. Ballhaus, A. Jameson, and J. Albert, "Implicit Approximate-Factorization Schemes for the Efficient Solution of Steady Transonic Flow Problems," NASA TM X-73202 (January 1977).

(26) J. C. South and A. Brandt, "Application of the Multi-Level Grid Method to Transonic Flows," Project SQUID Workshop on Transonic Flow Problems in Turbomachinery, Monterey, California, February 1976.

(27) G. Moretti, "On the Matter of Shock Fitting," Proceedings of the Fourth International Conference on Numerical Methods in Fluid Dynamics, Boulder, June 1974, Springer Verlag, New York, pp. 287-292 (1975).

(28) M. M. Hafez and H. K. Cheng, "Convergence Acceleration of Relaxation Solution for Transonic Flow Computations," AIAA J., V. 15, no. 3, pp. 329-336 (March 1977).

(29) N. J. Yu and A. R. Seebass, "Second Order Numerical Solutions of Transonic Flows Over Airfoils With and Without Shock Fitting," Symposium Transsonicum II, Gotingen, September 1975.

(30) A. Rizzi, "Transonic Solutions of the Euler Equations by the Finite Volume Method," Symposium Transsonicum II, Gottingen, September 1975.

(31) A. Jameson and D. A. Caughey, "Finite Volume Method for Transonic Potential Flow Calculations," AIAA Third Computational Fluid Dynamics Conference, New Mexico, June 1977.

(32) Y. H. Argyris and G. Mareczek, "Potential Flow Analysis by Finite Elements," Ingenieur-Archiv (January 1973).

(33) J. Periaux, "Three Dimensional Analysis of Compressible Potential Flows With the Finite Element Method," International Journal for Numerical Methods in Engineering, V. 9, pp. 775-831 (1975).

(34) L. C. Wellford, Jr. and J. T. Oden, "On Some Finite Element Methods for Certain Nonlinear Second Order Hyperbolic Equations," Ticom Report 74-7, The University of Texas at Austin (October 1974).

(35) M. Kawahara, "Steady and Unsteady Flow Analysis of Fluid by the Finite Element Method," Second International Symposium on Finite Element Methods in Flow Problems, June 1976.

(36) L. B. Wahlbin, "A Dissipative Galerkin Method for the Numerical Solution of First Order Hyperbolic Equations," in Mathematical Aspects of Finite Elements in Partial Differential Equations, ed. C. Boor, Academic Press, New York (1974).

(37) L. C. Wellford and M. M. Hafez, "An Implicit Velocity Formulation for the Small Disturbance Transonic Flow Problem Using Finite Elements," to be presented at the Symposium on Applications of Computer Methods in Engineering, University of Southern California, August 1977.

(38) A. Carasso, "A Least Squares Procedure for the Wave Equation," Mathematics of Computation, V. 28, no. 127, pp. 757-767 (July 1974).

(39) R. M. Beam and R. F. Warming, "An Implicit Finite-Difference Algorithm for Hyperbolic Systems in Conservation Law Form," J. of Comp. Physics, V. 22, no. 1, pp. 87-110 (September 1976).

(40) W. J. Phares and K. R. Kneile, "Solution to the Eulerian Equations by the Finite Element Method With an Application to Transonic Flow," Arnold Eng. Development Center (ARO Inc.) AEDC-TR-76-86.

(41) S. T. K. Chan and M. R. Brashears, "Finite Element Analysis of Transonic Flow," AFFDL-TR-74-11 (March 1974).

(42) J. Fix, "A Mixed Finite Element Scheme for Transonic Flows," I Case Report No. 76-25 (August 1976).

(43) P. Lesaint, "Continuous and Discontinuous Finite Element Methods for Solving the Transport Equation," Mathematics of Finite Elements and Applications, Brussel University (April 1975).

(44) L. C. Wellford and M. M. Hafez, "Mixed Finite Element Models and Dual Iterative Methods for Transonic Flow," Second International Symposium on Finite Element Methods in Flow Problems, Santa Margherita Ligure, Italy, June 1976.

(45) A. Jameson - private communication

(46) R. Glowinski and O. Pironneau, "On the Computation of Transonic Flows," France-Japanese Conference on Functional Analysis and Numerical Analysis, Tokyo, Kyoto, September 1976.

(47) R. Glowinski, J. Periaux, and O. Pironneau, "Transonic Flow Simulation by the Finite Element Method Via Optimal Control," International Symposium on Finite Element Methods in Flow Problems, Santa Margherita Ligure, Italy, June 1976.

(48) R. B. Fost, J. T. Oden, and L. C. Wellford, Jr., "A Finite Element Analysis of Shocks and Finite-Amplitude Waves in One-Dimensional Hyperelastic Bodies at Finite Strain," Int. J. Solids Structures, V. 11, pp. 377-401 (1975).

(49) J. T. Oden and S. N. Reddy, Mathematical Theory of Finite Elements, Wiley-Interscience, New York (1976).

(50) M. M. Hafez and E. M. Murman, "A Shock-Fitting Algorithm for the Full Potential Equation," Proceedings of the Third AIAA Computational Fluid Dynamics Conference, New Mexico, June 1977.

Key Words

- Transonic Flows, Mixed-Type Equations, Type-Sensitive Finite Difference Schemes
- Finite Volume
- Finite Element
- Least-Squares and Optimal Control Methods
- Entropy Constraints, Shock Capturing, Shock Fitting
- Generalized Relaxation and Steepest Descent (gradient, conjugate gradient, etc.) Methods

FIGURE CAPTIONS

Figure 1 Transonic Flow Pattern for a Lifting Airfoil (Nieuwland and Spee (1))

Figure 2 Shock-Free Airfoil (Bauer, et al. (11))

Figure 3 Lifting Shockless Airfoil (Boerstoel and Uijlenhoet (12))

Figure 4 Pressure Distribution for a Parabolic Arc Airfoil, K = 1.8. ($K = (1 - M_\infty^2)/M_\infty^2 \tau^{2/3}$, $\bar{C}_p = \tau^{2/3} C_p / M_\infty^{3/4}$) from Murman (17))

Figure 5a Bow Waves, Sonic Lines, and Pressure Coefficients for Supersonic Flows Past a NACA 0012 Airfoil (Rizzi (30))

Figure 5b Comparison Between Theory and Experiment for Flow Over a 6%-Thick Parabolic Arc at $M_\infty = 0.909$ and $\delta = 1.4$ (Rizzi (30))

Figure 6 Finite Volume Calculation (Jameson and Caughey (31))

Figure 7 Comparison Between Finite Element and Finite Difference Transonic Solutions (Wellford and Hafez (37))

Figure 8 Comparison of Chordwise Pressure Distribution for Supercritical Flow Over a 6-Percent Biconvex Airfoil at $M_\infty = 0.903$ (Phares and Kneile (40))

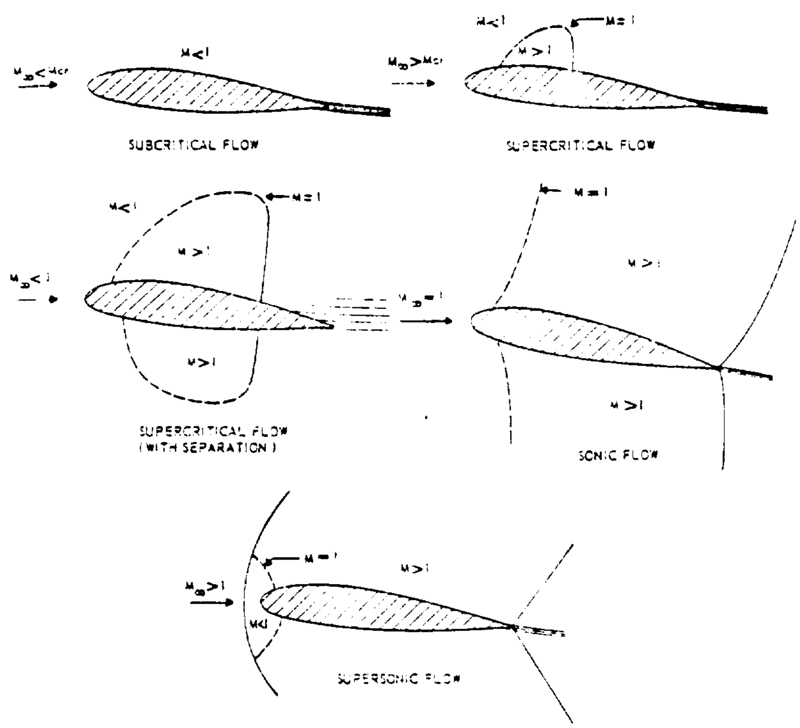
Figure 9 Comparison of Finite Element and Finite Difference Results for Transonic Flow (Hafez, et al. (21))

Figure 10 Comparison Between the Finite Element Solution of Transonic Flow and the Finite Difference Solution (Wellford and Hafez (44))

Figure 11a Mach Number Distribution on the Skin of a NACA 0012 Profile: k = 1, While Dots Are the Computational Results of Garabedian-Korn; k is the degree of polynomial approximation (Glowinski and Pironneau (46))

Figure 11b Same as Figure 11a, but for k = 2. Note That Three Small Reverse Shocks Are Still Present (Glowinski and Pironneau (46))

Figure 12 Comparison of Chordwise Pressure Distributions for NACA 64 A006 Airfoil (α = 0 deg) from Chan and Brashears (41)



SUBCRITICAL FLOW
SUPERCRITICAL FLOW
SUPERCRITICAL FLOW
(WITH SEPARATION)
SONIC FLOW
SUPERSONIC FLOW

Figure 1

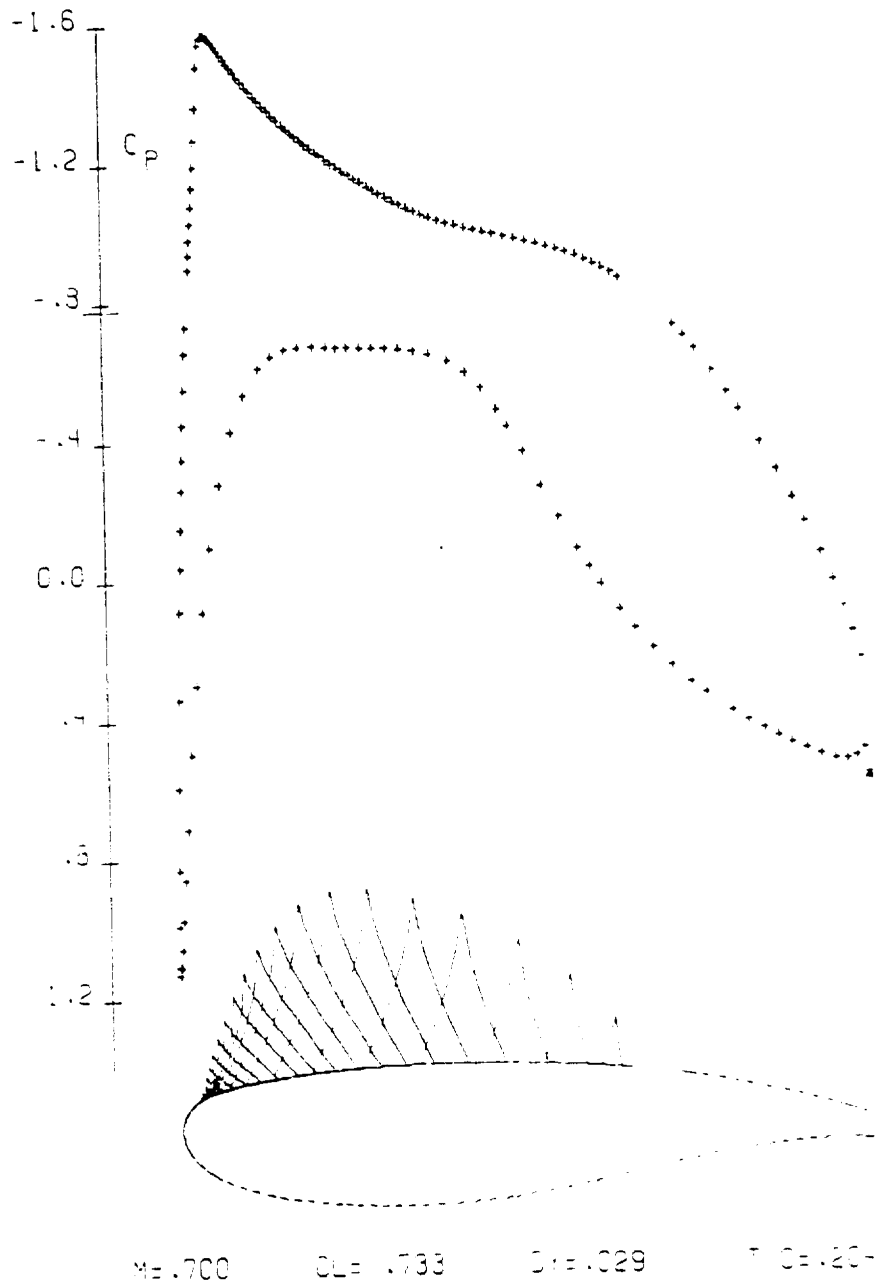


Figure 2

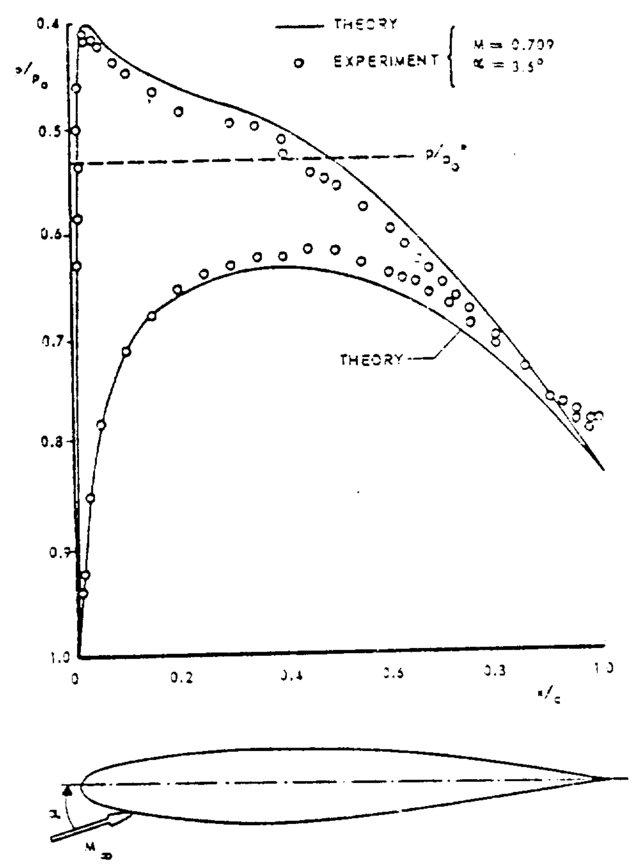

THEORY
EXPERIMENT
M = 0.709
α = 3.5°
p/p0
p/p0*
0.4
0.5
0.6
0.7
0.8
0.9
1.0
THEORY
0
0.2
1.0
x/c

Figure 3

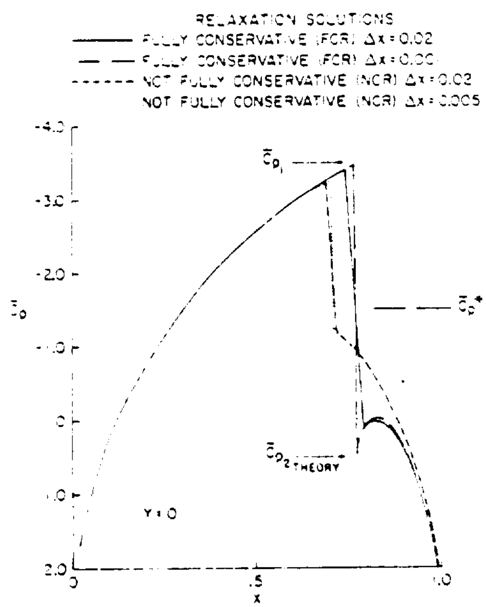

RELAXATION SOLUTIONS
FULLY CONSERVATIVE (FCR) Δx = 0.02
NOT FULLY CONSERVATIVE (NCR) Δx = 0.02
NOT FULLY CONSERVATIVE (NCR) Δx = 0.005
-4.0
-3.0
-2.0
0
2.0
.5
1.0
x
Y = 0

Figure 4

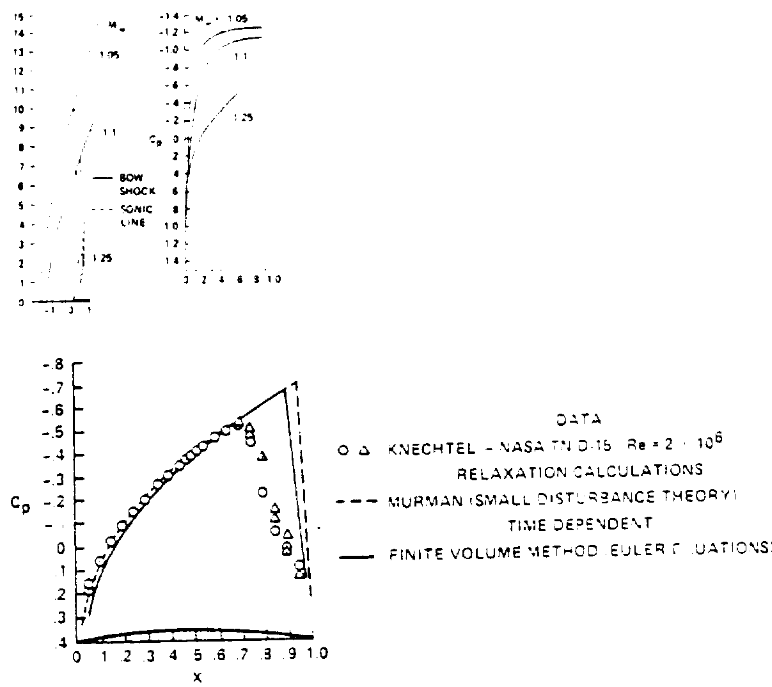

BOW SHOCK
SONIC LINE
DATA
RELAXATION CALCULATIONS
TIME DEPENDENT

Figure 5

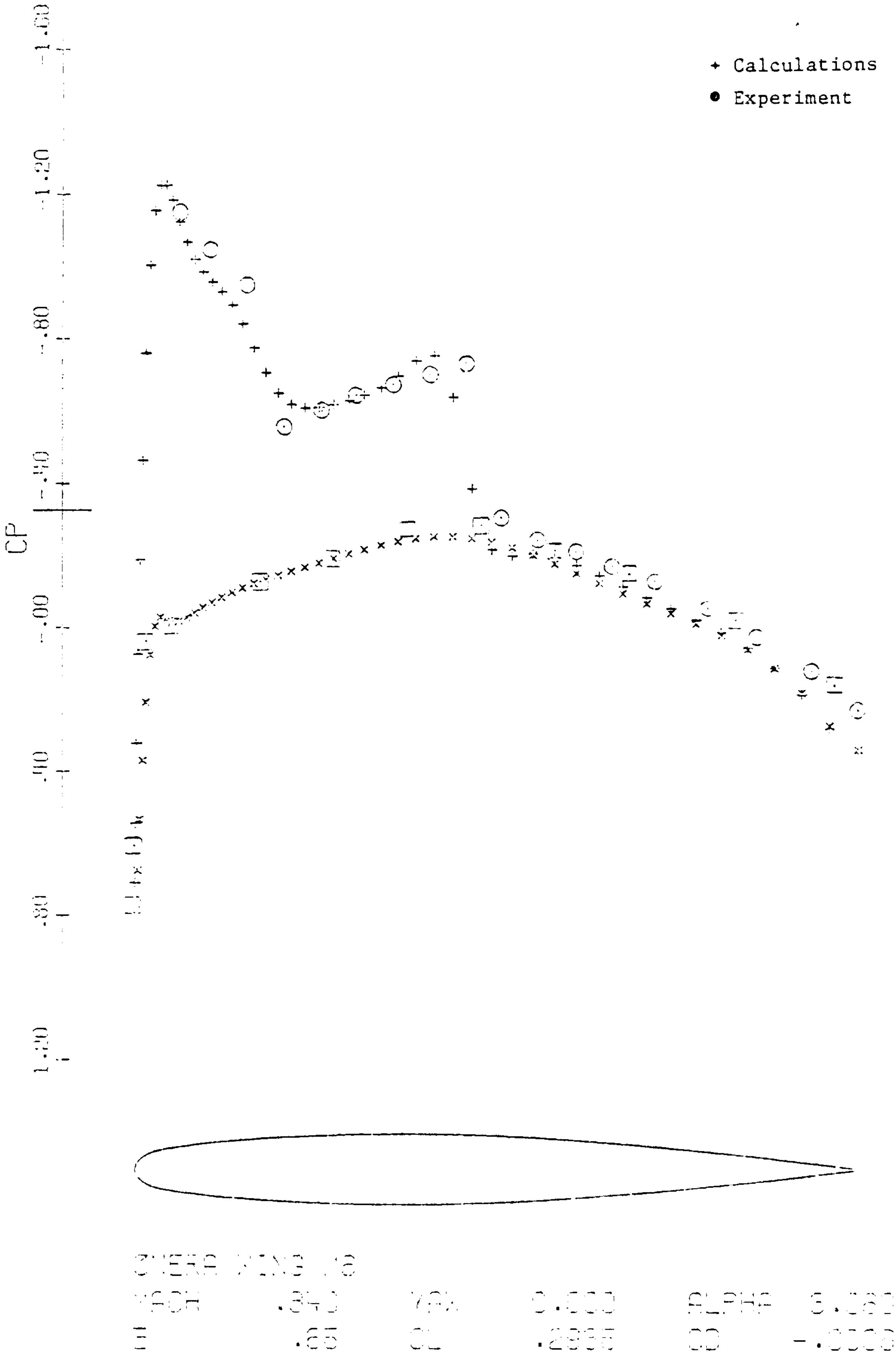
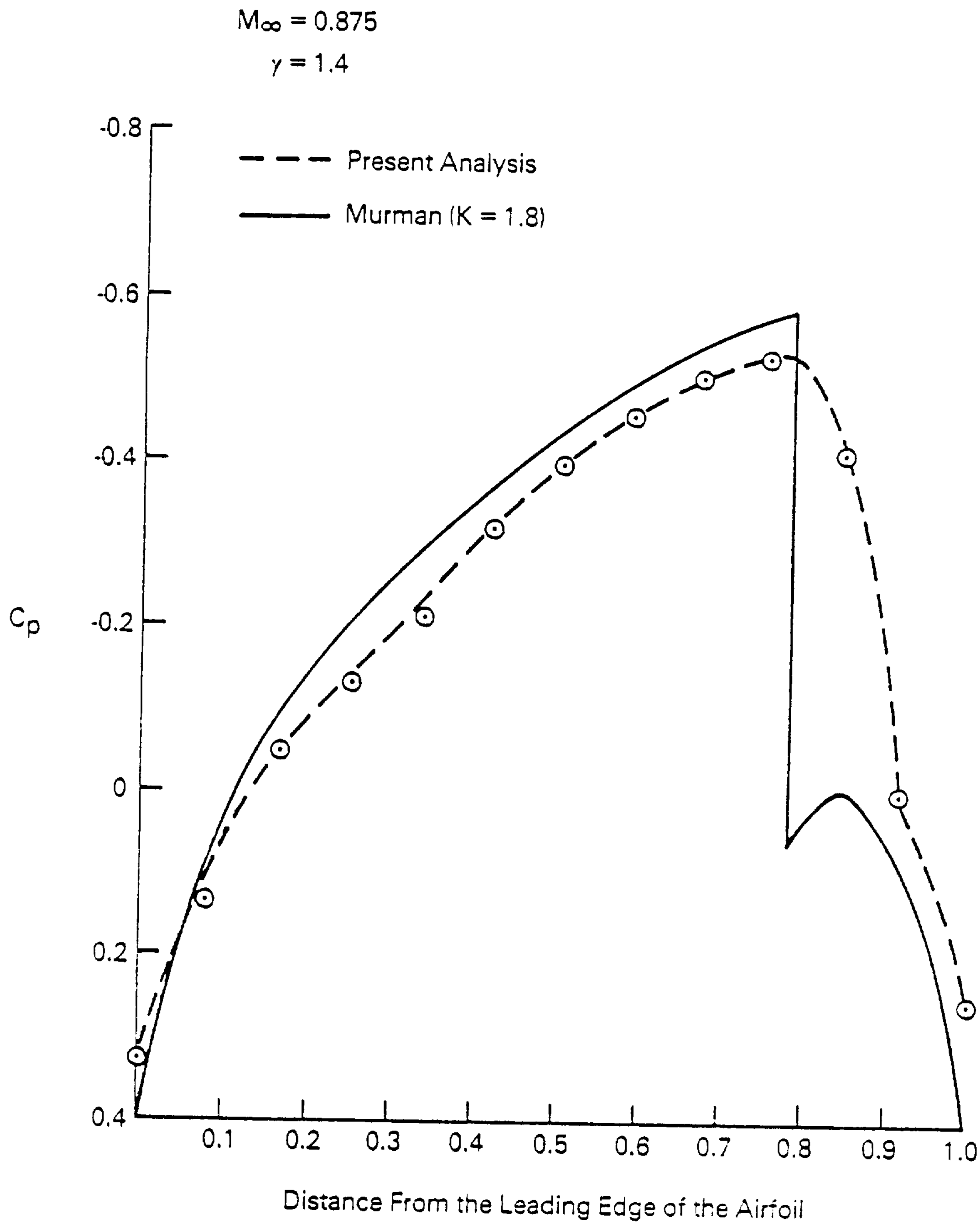


Figure 6



M∞ = 0.875
γ = 1.4
Present Analysis
Murman (K = 1.8)
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
Cp
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
Distance From the Leading Edge of the Airfoil

Figure 7

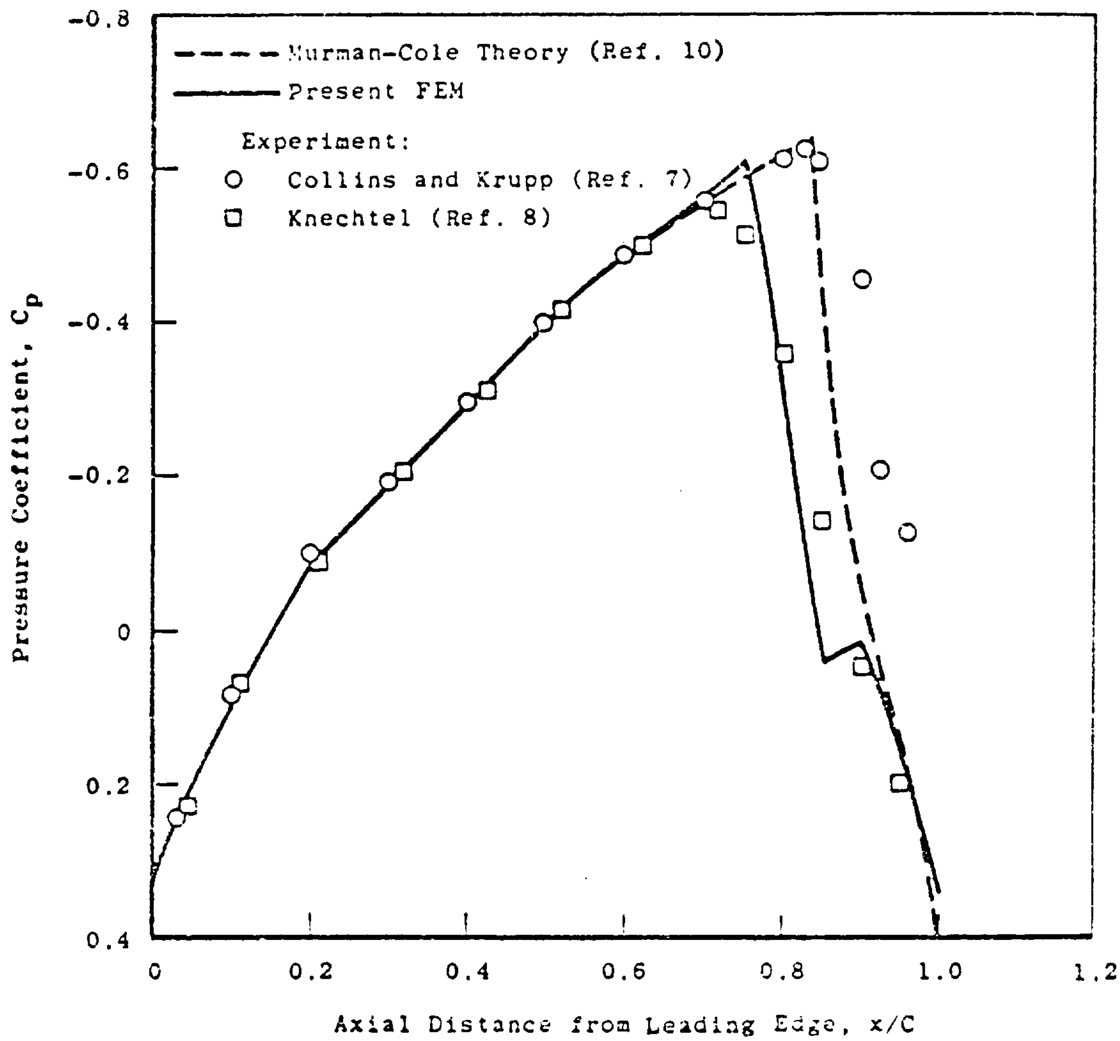

Murman-Cole Theory (Ref. 10)
Present FEM
Experiment:
Collins and Krupp (Ref. 7)
Knechtel (Ref. 8)
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
Pressure Coefficient, C_p
0
0.2
0.4
0.6
0.8
1.0
1.2
Axial Distance from Leading Edge, x/C

Figure 8

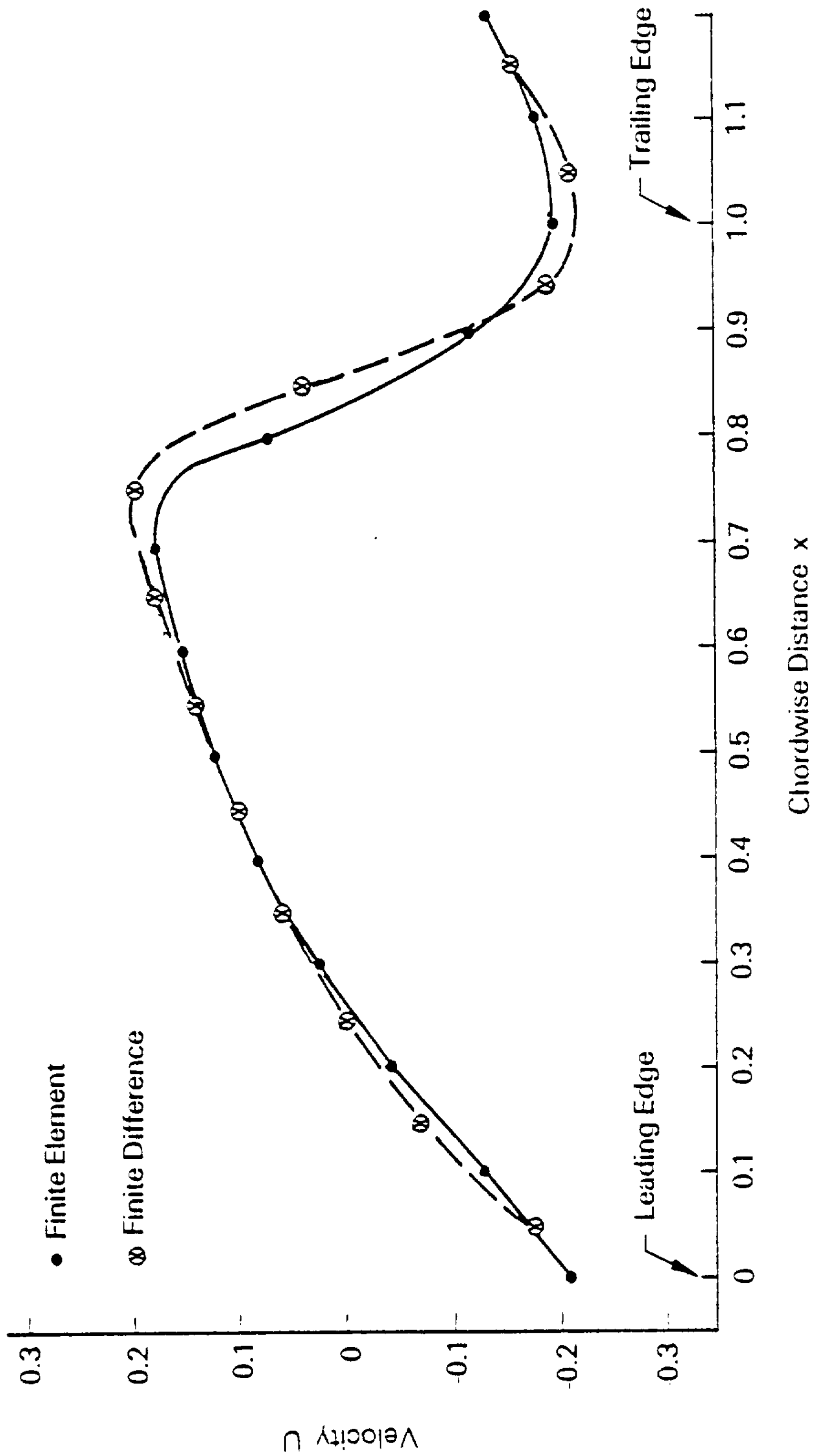
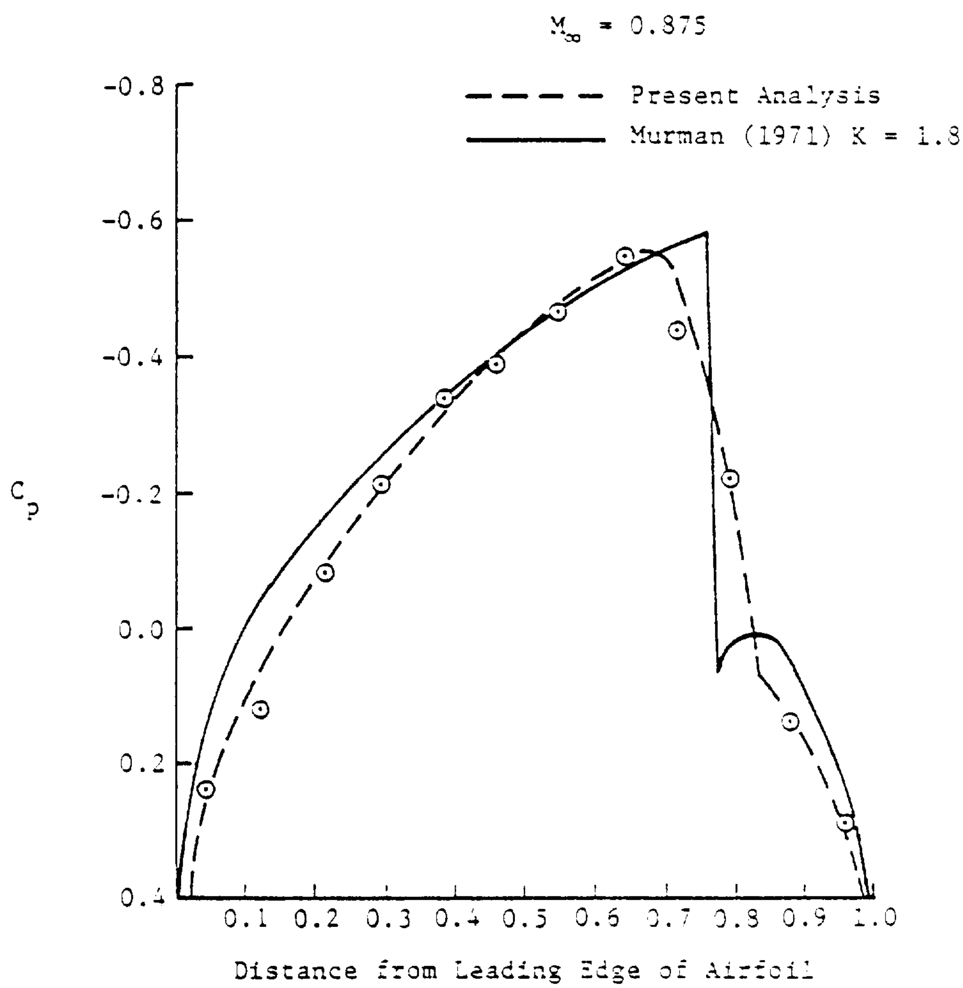

Finite Element
Finite Difference
Leading Edge
Trailing Edge
Chordwise Distance x
Velocity U
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1

Figure 9



M_∞ = 0.875
Present Analysis
Murman (1971) K = 1.8
-0.8
-0.6
-0.4
-0.2
0.0
0.2
0.4
C_p
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
Distance from Leading Edge of Airfoil

Figure 10

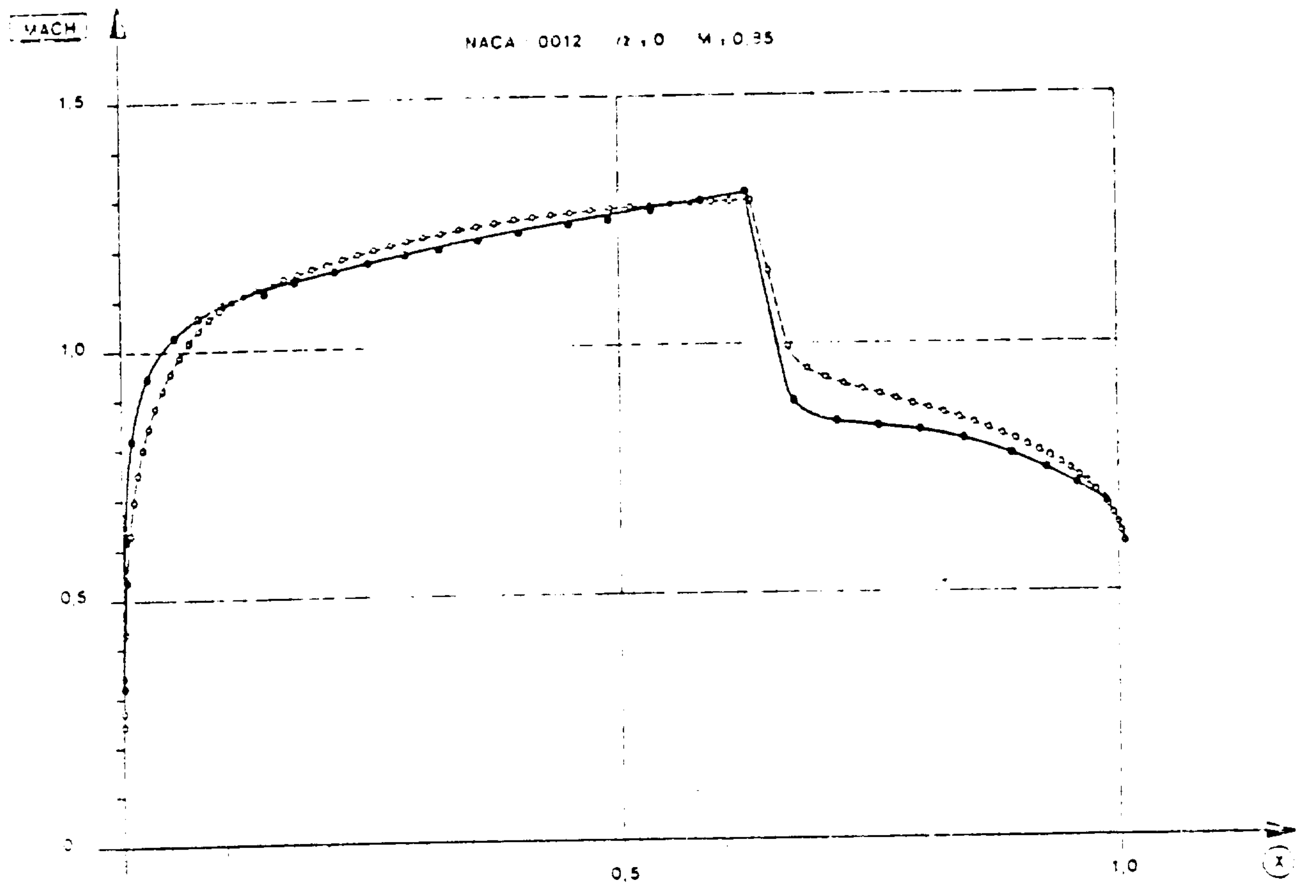


Figure 11

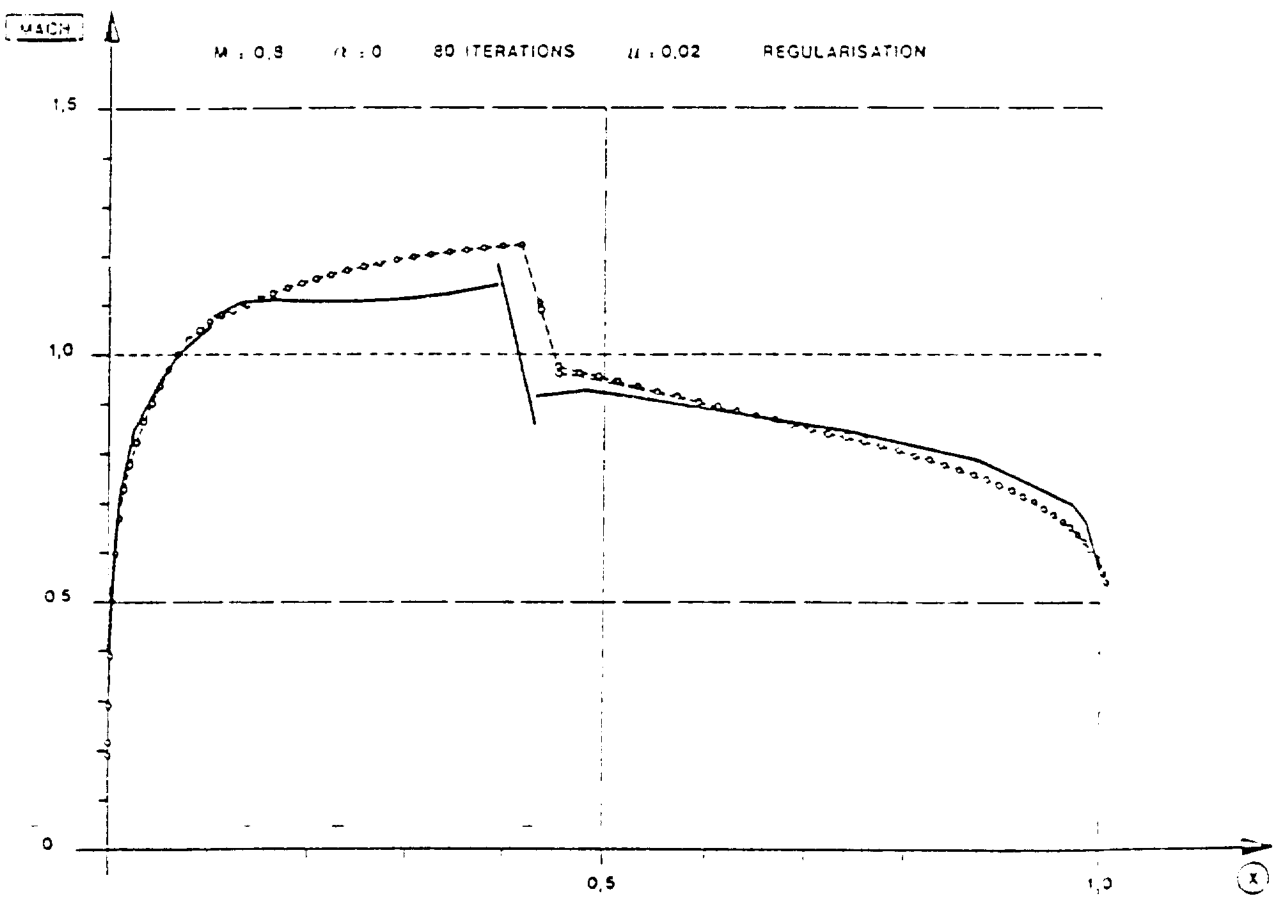
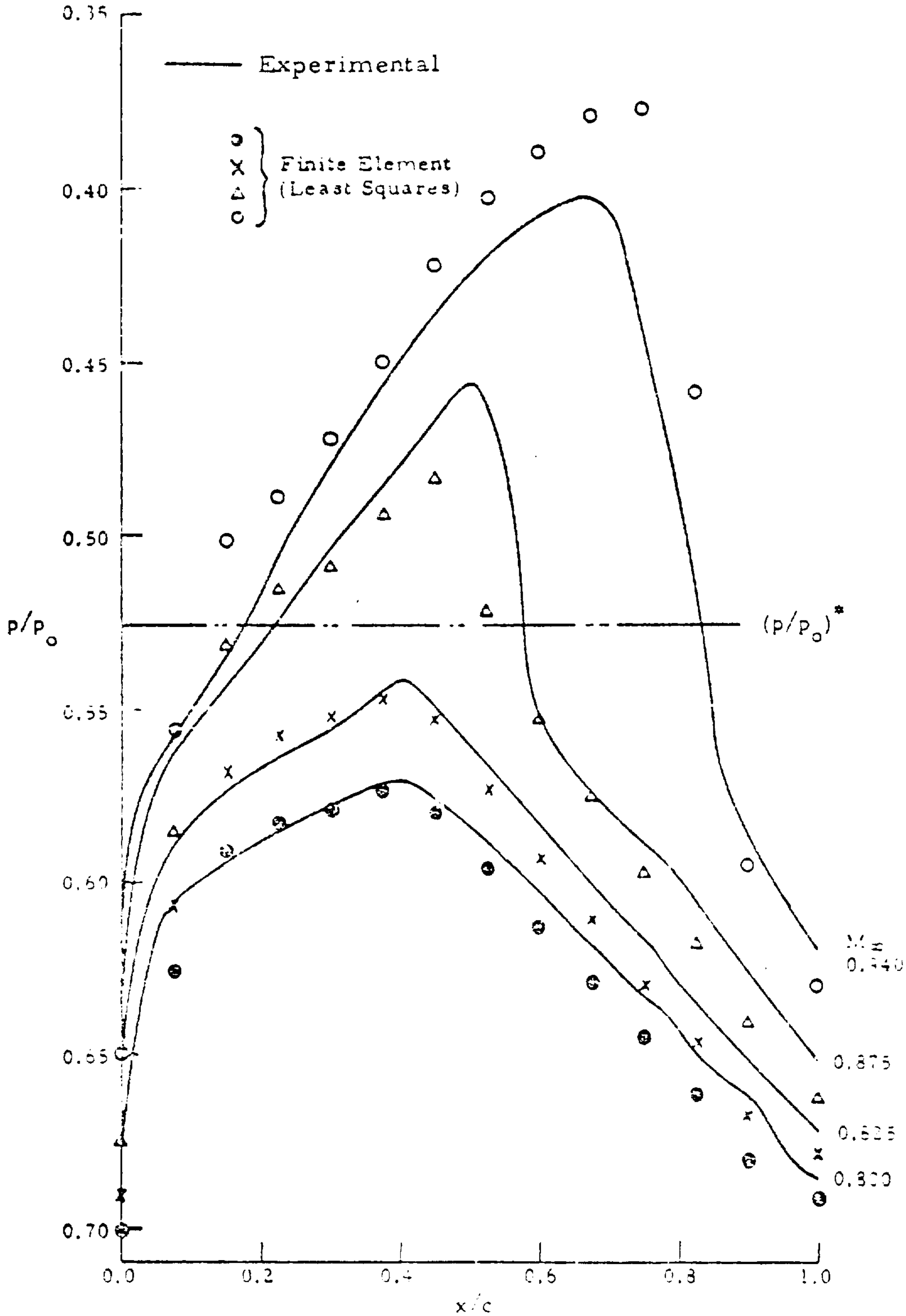

MACH
M : 0,8
α : 0
80 ITERATIONS
μ : 0,02
REGULARISATION
1,5
1,0
0,5
0
0,5
1,0
X

Figure 12



0.35
0.40
0.45
0.50
0.55
0.60
0.65
0.70
Experimental
Finite Element
(Least Squares)
p/p$_o$
(p/p$_o$)*
M =
0.840
0.875
0.825
0.800
0.0
0.2
0.4
0.6
0.8
1.0
x/c

Figure 13